(10) 1. Suppose $f(x) = 5 - 3\sin(4x) + 7\cos(9x)$ on the interval $[-\pi, \pi]$.

a) What is the Fourier series of $f(x)$ on the interval $[-\pi, \pi]$?

b) What is $\displaystyle\int_{-\pi}^{\pi} (f(x))^2\, dx$?

(12) 2. Suppose $f(x) = \pi x^2 - 2x^3$ on $[0, \pi]$, and $g(x)$ is the sum of the whole Fourier sine series for $f(x)$, and $h(x)$ is the sum of the whole Fourier cosine series for $f(x)$.

a) Compute these numbers.

$f(1) =$ ____. $g(1) =$ ____. $h(1) =$ ____. $f(\pi) =$ ____. $g(\pi) =$ ____. $h(\pi) =$ ____. $g(-\frac{\pi}{2}) =$ ____. $h(-\frac{\pi}{2}) =$ ____.

b) Is $g(x)$ continuous on $[-\pi, \pi]$? ____. (Just **Yes** or **No**: no explanation is necessary.)

c) Is $h(x)$ continuous on $[-\pi, \pi]$? ____. (Just **Yes** or **No**: no explanation is necessary.)

(16) 3. Suppose $f(x) = x^2$.

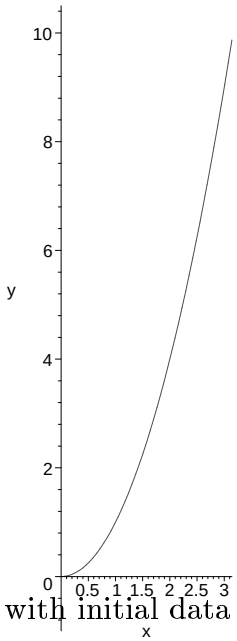
a) Compute $\displaystyle\int f(x)\sin(nx)\, dx$.

Comment Yes, this is an indefinite integral. Yes, you should integrate by parts twice. Yes, you can *guess* the answer, but then you must verify the answer by differentiation.

b) Compute $m_n = \displaystyle\int_0^{\pi} f(x)\sin(nx)\, dx$ as explicitly as you can when n is a positive integer.

c) Give exact values for m_1 and m_2 and m_3 and m_4.

d) Suppose $g(x) = \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{100} m_n \sin(nx)$. A graph of $f(x) = x^2$ for x in $[0, \pi]$ is given. Sketch a reasonable approximation to $g(x)$ on this graph.

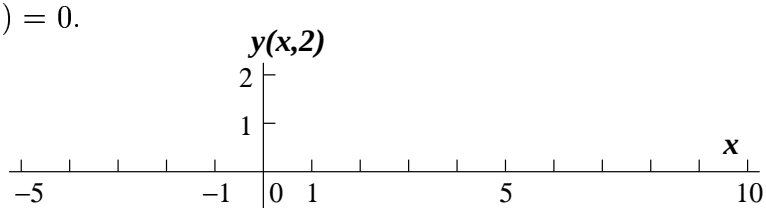


Comment Yes, there *is* a relationship between $g(x)$ and $f(x)$.

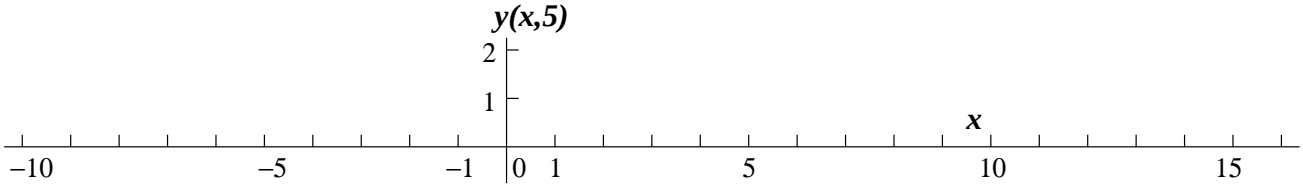
(16) 4. Consider the wave equation $\dfrac{\partial^2 y}{\partial t^2} = 4\dfrac{\partial^2 y}{\partial x^2}$ (here $c = 2$) for x in all of $\mathbb{R}$ with initial data $y(x, 0) = 4e^{-(x-3)^2}$ and $\dfrac{\partial y}{\partial t}(x, 0) = 0$.

a) Find $y(x, t)$.

b) Sketch $y(x, 2)$.

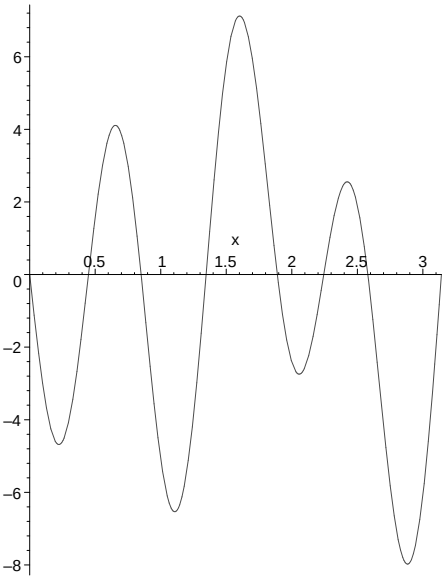


c) Sketch $y(x, 5)$.



d) At approximately what positive time will the displacement at $x = 30$ be equal to 2?

(14) 5. Consider the wave equation $\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial^2 y}{\partial x^2}$ (here $c = 1$) for x in $[0, \pi]$ with boundary conditions $\begin{cases} y(0,t) = 0 \\ y(\pi,t) = 0 \end{cases}$ and initial conditions $\begin{cases} y(x,0) = 2\sin(4x) - 2\sin(3x) - 5\sin(7x) \\ \dfrac{\partial y(x,0)}{\partial t} = 0 \end{cases}$.



a) Write a formula for $y(x,t)$.

b) A `Maple` graph of $y(x,0)$ is shown to the right.

i) What does a graph of $y(x, 2\pi)$ look like (compared to this graph) and why? *Use complete English sentences.*

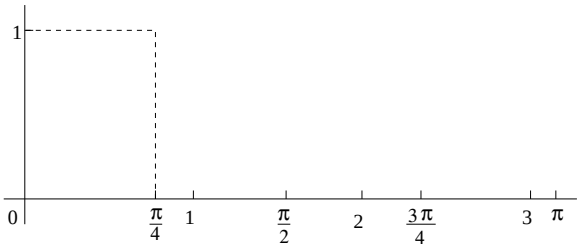
ii) Certainly $|y(x,t)| \leq 10^{10}$ for all x in $[0,\pi]$ and all t. Give a much lower overestimate for the largest possible displacement at any time. Be sure to give evidence supporting your assertion.

(16) 6. Suppose a bar of length π placed on the interval $[0,\pi]$ has insulated sides *and* ends. An initial temperature distribution $u(x,0)$ is given: $u(x,0) = \begin{cases} 1 & \text{if } x < \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$. The temperature distribution satisfies the heat equation $\dfrac{\partial^2 u}{\partial x^2}(x,t) = \dfrac{\partial u}{\partial t}(x,t)$ (here $k = 1$) for all x in $[0,\pi]$ and $t \geq 0$.

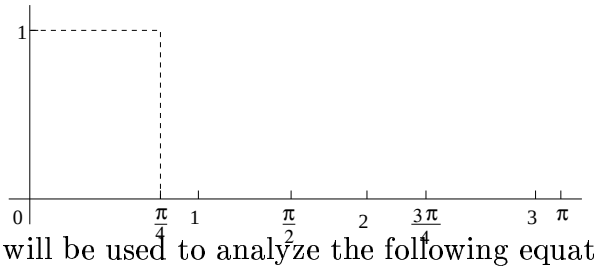
a) Find $u(x,t)$ as well as you can ($u(x,t)$ will be the sum of an infinite series).

b) Write the first 5 non-zero terms of the series for $u(x,t)$ as explicitly as you can.

c) Sketch a graph of $u(x, \frac{1}{100})$.



d) Sketch a graph of $u(x, 100)$.



(16) 7. In this problem, separation of variables will be used to analyze the following equation:

$$(*) \qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u$$

a) If $u(x,t) = X(x)T(t)$, find ordinary differential equations which $X(x)$ and $T(t)$ must satisfy if $u(x,t)$ is a solution of $(*)$.

b) Suppose the solution $u(x,t)$ found in a) also satisfies the boundary conditions $u(0,t) = 0$ and $u(\pi,t) = 0$ for all t. How are $X(x)$ and $T(t)$ further restricted?

c) Use your asnwer to b) to write a formula (it will be an infinite series) for the most general solution to $(*)$ which satisfies the boundary conditions $u(x,0) = 0$ and $u(x,\pi) = 0$.

Third Exam for Math 421, section 2

May 11, 2004

NAME _____

Do all problems, in any order.
Show your work. An answer alone may not receive full credit.
No notes other than the distributed formula sheet may be used on this exam.
No calculators may be used on this exam.

Problem Number	Possible Points	Points Earned:
1	10	
2	12	
3	16	
4	16	
5	14	
6	16	
7	16	
Total Points Earned:		