Sample problems for exam #2 in Math 421

1. Suppose $A = (a_{ij})$ is a 200 by 200 matrix. *All of the entries of A are* 0 <u>except</u>

Diagonals Each $a_{ii} = 2$ for $1 \le i \le 200$.

Weirdos $a_{ij} = 10$ when $i = 100$ and $j = 150$, and $a_{ij} = x$ when $i = 150$ and $j = 30$.

What is the determinant of A?

Comment A matrix with relatively few non-zero entries is called a *sparse matrix.*

2. Suppose this matrix is in RREF: $\begin{pmatrix} \star & 7 & \star & \star & \star & \star & \star & \star \\ \star & \star & 0 & 1 & \star & \star & \star & \star \\ \star & \star & \star & \star & 0 & 0 & 0 & 0 \end{pmatrix}$. Change each $\star$ to one of the following: **0** if the entry *must* be 0; **1** if it *must* be 1; **A** if it can be *anything* (0 or $\neq 0$: no restrictions). Explain your answers. What's the rank of this matrix?

3. Suppose $(1,2,3,0,0)$ and $(0,0,1,1,1)$ are both solutions of one specific system of 5 linear equations in 5 unknowns.

a) If this system is written as $AX = B$ where A is a matrix of coefficients, X is a column vector of unknowns, and B is a column vector of specific real numbers (you can't deduce much about the entries of A and B from what's given), then what are the dimensions of A and X and B?

b) Does the system $AX = 0$ (here 0 is a column vector of 0's) have any solutions? Be as specific about the solutions as you can.

c) How many solutions does the original system $AX = B$ have? Be as specific about the solutions as you can.

d) What is the determinant of A? What are the possible values of the rank of A?

4. Suppose that A and B are both 2 by 2 matrices which have inverses, and that $A^{-1} = \begin{pmatrix} 3 & -1 \\ -8 & 1 \end{pmatrix}$ and $B^{-1} = \begin{pmatrix} -2 & 1 \\ -1 & 7 \end{pmatrix}$. If $C = A^2 B$, what is the inverse of C?

Comment A small amount of thought can save much irritation.

5. Suppose that A is a 50 by 70 matrix whose rank is 40. What is the dimension of the subspace S of $\mathbb{R}^N$ consisting of the solutions of the homogeneous system $AX = 0$? What is the dimension of the subspace S of $\mathbb{R}^M$ consisting of all possible Y's for which the equation $AX = Y$ can be solved? What *are* N and M?

6. Find all values of x so that the matrix $\begin{pmatrix} x & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ x & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$ has an inverse.

7. The matrix $\begin{pmatrix} 2 & u & v & w \\ 0 & -1 & 0 & s \\ 0 & 0 & 3 & t \\ 0 & 0 & 0 & 4 \end{pmatrix}$ is invertible for all u, v, w, s, and t. What is its inverse?

Comment This matrix is in *upper-triangular form* with an upper-triangular inverse. First use row operations to make the diagonal elements $= 1$, then pivot and eliminate.

TREE 8. Are $(-2,-4,8,-7)$, $(1,2,1,1)$, and $(1,2,-1,2)$ linearly independent?

9. Is the sum of invertible matrices always invertible? Is the sum of singular matrices always singular?

OVER

TREE 10. What is the rank of $A = \begin{pmatrix} 0 & 2 & 5 & 2 & 0 \\ 2 & -1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & -1 & -1 & -1 & 0 \end{pmatrix}$? Find a basis for the subspace of all solutions of $AX = 0$ in $\mathbb{R}^P$. What is P?

11. Briefly explain why the vectors $(3, 4, -1)$, $(2, 2, 1)$, $(1, 2, 0)$, and $(0, 1, -4)$ in $\mathbb{R}^3$ must be linearly dependent. Do this in two ways:

i) Use a very short argument based on *dimension.*

ii) Use the fact that a specific homogeneous system *must* have a non-trivial solution.

TREE 12. Verify that $(1, 2, 0, 2, 1)$ is in the linear span of $(1, 1, 0, 0, 2)$ and $(2, 1, 2, -2, 1)$ and $(4, 2, 5, -4, 0)$.

13. Suppose A is a 3 by 3 matrix, and $\det(A) = 2$. Compute $\det(A^5)$ and $\det(5A)$ and $\det(A^{-1})$ and $\det(AA^t)$.

14. Be prepared to discuss the meaning of ... HOMOGENEOUS ... INHOMOGENEOUS ... BASIS ... LINEAR INDEPENDENCE ... LINEAR COMBINATION ... SUBSPACE ... DIMENSION ... SPANNING ... RANK ... EIGENVALUE ... EIGENVECTOR ... TRANSPOSE ... MATRIX ADDITION AND MULTIPLICATION ... SYMMETRIC ... DIAGONALIZATION ... LIFE.

TREE 15. Consider the system $\begin{cases} 3x_1 + 5x_2 - 1x_3 = a \\ 2x_1 + 6x_2 + 1x_3 = b \\ 4x_1 + 4x_2 - 3x_3 = c \end{cases}$.

a) Find a specific vector (a, b, c) in $\mathbb{R}^3$ so that the system has *no* solution. Explain.

b) Find a specific vector (a, b, c) in $\mathbb{R}^3$ so that the system has *at least one* solution. Find all solutions for that vector. Explain.

16. Suppose A is a 4 by 4 matrix whose <u>inverse</u> is $\begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 3 \\ 4 & 1 & 2 & 2 \end{pmatrix}$.

a) Find all solutions of $AX = B$ if $B = (2, 1, 1, 0)$.

b) Find all solutions of the homogeneous system $AX = 0$.

17. Give an example of a specific matrix which has exactly 20 non-zero entries and whose characteristic polynomial is $(\lambda - 5)^3(\lambda + 2)^5(\lambda^2 + 4)$.

TREE 18. a) Explain why the vectors $(2, -3, 5)$, $(-7, 1, -2)$, and $(3, -5, 7)$ are a basis of $\mathbb{R}^3$.

b) Write a linear combination of the vectors in a) whose sum is equal to the vector $(2, 1, 3)$.

19. a) Diagonalize $A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$. That is, find the eigenvalues and eigenvectors of A, and find a matrix T so that TAT^{-1} is diagonal.

b) $3^5 = 243$. Use this fact and the diagonalization of A found in a) to compute A^5.

20. If $A = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$, then A has eigenvalues -2, 0, 1, and 3 with associated eigenvectors $(0, 0, 0, 1)$, $(-1, 1, 1, 0)$, $(0, -1, 1, 0)$, and $(2, 1, 1, 0)$, respectively. Find an orthogonal matrix W so that WAW^{-1} is diagonal.

21. Explain why $A = \begin{pmatrix} 0 & 0 \\ 231 & 0 \end{pmatrix}$ *can't* be diagonalized: that is, there is no invertible matrix T so that TAT^{-1} is diagonal.

New Jersey native trees: some matrices and their reduced row echelon form

PINE (5 by 4) $\begin{pmatrix} 1 & 2 & 4 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & 2 & 5 & 0 \\ 0 & -2 & -4 & 2 \\ 2 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

BIRCH (4 by 5) $\begin{pmatrix} 1 & 1 & 0 & 0 & 2 \\ 2 & 1 & 2 & -2 & 1 \\ 4 & 2 & 5 & -4 & 0 \\ 1 & 2 & 0 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -2 & 3 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

SYCAMORE (4 by 3) $\begin{pmatrix} -2 & 1 & 1 \\ -4 & 2 & 2 \\ 8 & 1 & -1 \\ -7 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & \frac{3}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

HICKORY (3 by 4) $\begin{pmatrix} -2 & -4 & 8 & -7 \\ 1 & 2 & 1 & 1 \\ 1 & 2 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$

OAK (4 by 5) $\begin{pmatrix} 0 & 2 & 5 & 2 & 0 \\ 2 & -1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & -1 & -1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -3 & -3 \\ 0 & 1 & 0 & -4 & -5 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

CEDAR (5 by 4) $\begin{pmatrix} 0 & 2 & 1 & 1 \\ 2 & -1 & 0 & -1 \\ 5 & 1 & 2 & -1 \\ 2 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

WILLOW (3 by 4) $\left(\begin{array}{ccc|c} 3 & 5 & -1 & a \\ 2 & 6 & 1 & b \\ 4 & 4 & -3 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -\frac{11}{8} & \frac{3}{4}a - \frac{5}{8}b \\ 0 & 1 & \frac{5}{8} & -\frac{1}{4}a + \frac{3}{8}b \\ 0 & 0 & 0 & -2a + b + c \end{array}\right)$

ALDER (3 by 4) $\left(\begin{array}{ccc|c} 3 & 2 & 4 & a \\ 5 & 6 & 4 & b \\ -1 & 1 & -3 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & \frac{3}{4}a - \frac{1}{4}b \\ 0 & 1 & -1 & -\frac{5}{8}a + \frac{3}{8}b \\ 0 & 0 & 0 & \frac{11}{8}a - \frac{5}{8}b + c \end{array}\right)$

REDBUD (3 by 4) $\left(\begin{array}{ccc|c} 2 & -3 & 5 & a \\ -7 & 1 & -2 & b \\ 3 & -5 & 7 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{3}{25}a - \frac{4}{25}b + \frac{1}{25}c \\ 0 & 1 & 0 & \frac{43}{25}a - \frac{1}{25}b - \frac{31}{25}c \\ 0 & 0 & 1 & \frac{32}{25}a + \frac{1}{25}b - \frac{19}{25}c \end{array}\right)$

ASH (3 by 4) $\left(\begin{array}{ccc|c} 2 & -7 & 3 & a \\ -3 & 1 & -5 & b \\ 5 & -2 & 7 & c \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{3}{25}a - \frac{43}{25}b + \frac{32}{25}c \\ 0 & 1 & 0 & -\frac{4}{25}a - \frac{1}{25}b + \frac{1}{25}c \\ 0 & 0 & 1 & \frac{1}{25}a - \frac{31}{25}b - \frac{19}{25}c \end{array}\right)$