Sample problems for exam #3 in Math 421:02

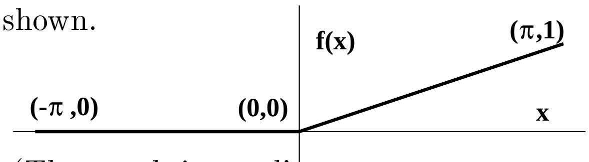
4/28/2004

Thanks to Professor N. Komarova for many of these problems. *Most* (but not all) of the problems are written so that NO INTEGRAL COMPUTATIONS ARE NEEDED!

1. If $f(x) = 3\cos(5x) - 9\sin(2x) + 8\sin(33x)$, compute $\int_{-\pi}^{\pi} (f(x))^2 \, dx$.

2. Use the complex form of sine and the formula for the sum of a geometric series to find a simple *real* formula for $\sum_{n=1}^{\infty} \frac{1}{2^n}\sin(nx)$.

3. Suppose $f(x)$ is the function with the graph shown.
a) Find the Fourier coefficients of $f(x)$.
b) Suppose $g(x)$ is the sum of the first 100 terms of the Fourier series of $f(x)$ (both sine and cosine). Draw a graph of both $y = g(x)$ and $y = f(x)$ on the interval $[0, 3]$. Draw a graph of both $y = g(x)$ and $y = f(x)$ on the interval $(3, \pi)$. Label the graphs.
c) Suppose now $h(x)$ is the sum of the **whole** Fourier series of $f(x)$. Graph $f(x)$ and $h(x)$ on the interval $[-\pi, \pi]$. Label the graphs.



(The graph is two line segments connecting the three indicated points.)

4. Calculate the Fourier series of the functions given (all defined on the interval $[-\pi, \pi]$):
a) $f(x) = -1$ if $x \le 0$ and 2 if $x > 0$. b) $f(x) = 5$. c) $f(x) = 21 + 2\sin(5x) + 8\cos(2x)$.
d) $f(x) = \sum_{n=1}^{8} c_n \sin(nx)$; $c_n = \frac{1}{n}$. e) $f(x) = -4 + \sum_{n=1}^{6} \big(c_n \sin(nx) + 7\cos(nx)\big)$; $c_n = (-1)^n$.

5. a) Suppose $f(x) = x + x^3$ for x in $[-\pi, \pi]$. Which coefficients of the Fourier series of $f(x)$ *must* be 0?

b) Suppose $f(x) = \cos(x^5) + \sin(x^2)$ for x in $[-\pi, \pi]$. Which coefficients of the Fourier series of $f(x)$ *must* be 0?

6. Suppose $f(x) = x + x^4$ for x in $[0, \pi]$.
a) If $F(x)$ is the odd extension of $f(x)$ to $[-\pi, \pi]$, write a formula or formulas for $F(x)$. Which terms *must* be 0 in the Fourier series of $F(x)$?
b) If $G(x)$ is the even extension of $f(x)$ to $[-\pi, \pi]$, write a formula or formulas for $G(x)$. Which terms *must* be 0 in the Fourier series of $G(x)$?

7. Suppose $f(x) = 2e^{-4x}$ for x in $[0, \pi]$. Another function, $F(x)$, is given by $F(x) = \sum_{n=0}^{\infty} b_n \sin(nx)$, where $b_n = \frac{2}{\pi}\int_0^{\pi} \left(2e^{-4x}\right)\sin(nx)\, dx$. Compute $F(3)$ and $F(-2)$ in terms of values of the exponential function.

8. Both ends of a string of length 25 cm are attached to fixed points at height 0. Initially, the string is at rest, and has the shape $4\sin\left(\frac{2\pi x}{25}\right)$, where x is the horizontal coordinate along the string, with 0 at the left end. The speed of wave propagation along the string is 3 cm/sec. Write the initial and boundary value problem for the shape of the string.

9. Suppose the following boundary value problem is given:
$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$ for x in $[0, \pi]$; $y(0, t) = y(\pi, t) = 0$; $y(x, 0) = 5\sin(2x) + 8\sin(6x)$; $\frac{\partial y(x,0)}{\partial t} = 0$.
Find $y(x, t)$.

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10. Suppose the following boundary value problem is given:

$$\frac{\partial^2 y}{\partial t^2} = 50\frac{\partial^2 y}{\partial x^2} \text{ for } x \text{ in } [0,100];\ y(0,t) = y(100,t) = 0;\ y(x,0) = x^2(100-x);$$
$$\frac{\partial y(x,0)}{\partial t} = x \text{ for } x \text{ in } [0,25] \text{ and } \tfrac{1}{3}(100-x) \text{ for } x \text{ in } (25,100].$$

What is the speed of wave propagation along the string? What is the initial displacement of the string at point $x = 20$? What is the initial velocity of the string at point $x = 50$? At what point of the string is the initial velocity the largest?

11. Suppose the following boundary value problem is given:

$$\frac{\partial^2 y}{\partial t^2} = 50\frac{\partial^2 y}{\partial x^2} \text{ for } x \text{ in } [0,\pi];\ y(0,t) = y(\pi,t) = 0;\ y(x,0) = 0;\ \frac{\partial y(x,0)}{\partial t} = g(x).$$

Suppose we also know $\int_0^\pi g(x)\sin(nx)\,dx = \frac{1}{n^3}$ for all positive integers, n. Find $y(x,t)$.

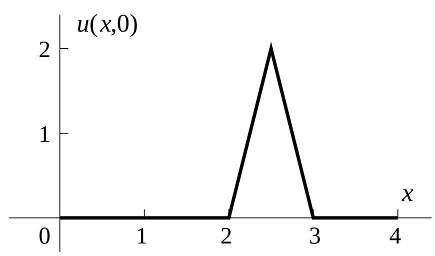
12. Use separation of variables to analyze the equation $\frac{\partial y}{\partial t} = 12y - 5\frac{\partial y}{\partial x} + 7\frac{\partial^2 y}{\partial x^2}$.
That is, reduce this partial differential equation to some ordinary differential equations. Explain every step.

13. Consider this wave equation and initial value problem on the whole real line, $\mathbb{R}$:

$$\frac{\partial^2 y}{\partial t^2} = 9\frac{\partial^2 y}{\partial x^2} \text{ for } x \text{ in } \mathbb{R};\ y(x,0) = x(2-x) \text{ for } x \text{ in } [0,2]\ 0 \text{ otherwise};\ \frac{\partial y(x,0)}{\partial t} = 0 \text{ for } x \text{ in } \mathbb{R}.$$

a) Find $y(x,t)$.
b) Draw the solution for $t = 5$ and $t = 10$ (two graphs).
c) How long will it take before an observer located at point $x = 27$ receives a signal?

14. The graph displays an initial ($t = 0$) temperature distribution for $u(x,t)$, the temperature on an insulated bar 4 cm long satisfying the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ and the boundary conditions $u(0,t) = 0$ and $u(4,t) = 0$ for all t.
a) Sketch a graph of $u(x, \frac{1}{100})$ and a graph of $u(x,100)$.
b) Explain why this initial temperature distribution can't result from an earlier temperature distribution.



15. Suppose an insulated bar of length π cm also has insulated ends. An initial temperature distribution is given by $f(x) = 2$ for x in $[0, \frac{\pi}{2}]$ and 4 for x in $(\frac{\pi}{2}, \pi]$.
a) Write the initial and boundary value problem for $y(x,t)$, the temperature of the bar.
b) Write the solution, $u(x,t)$.
c) What is the limiting temperature distribution as $t \to \infty$?
d) Draw the temperature distribution for $t = 0$, for some $t_1 > 0$, for some $t_2 > t_1$ and for $t = \infty$ (four graphs).

16. Suppose an insulated bar of length 10 cm has insulated ends. Find the temperature distribution as $t \to \infty$ if:
a) The initial temperature distribution is given by $f(x)$ where $f(x) = 0$ if x is in $[0,1]$, 2 if x is in $(1,2]$, 0 if x is in $(2,3]$, 5 if x is in $(3,4]$, and 2 if x is in $(4,6]$.
b) The initial temperature distribution is given by $f(x) = x + 2x^2$.

17. An insulated bar of length 5 cm has its left end kept at temperature 0 for all t, and its right end kept at temperature 5 for all t. The bar's initial temperature distribution is given by $f(x) = 6x - x^2$. If $u(x,t)$ is the temperature distribution at time t for $t \geq 0$, then draw the temperature distribution for $t = 0$, $t = \frac{1}{100}$, and $t = 100$ (three graphs).