

Math 421:03 Some two-dimensional problems December 5, 2004

The last two lectures will discuss some aspects of solutions of the heat and wave equations for two-dimensional regions. Sections 13.5 and 13.8 of the text contain some relevant material. In these problems, S denotes the π -by- π square with lower left corner at $(0,0)$.

1. In the lecture a solution to $\Delta u = u_{xx} + u_{yy} = 0$ on S was found which satisfied (BC) $u(x, 0) = 0$ & $u(x, \pi) = 0$ for $0 \leq x \leq \pi$; $u(0, y) = 0$ & $u(\pi, y) = 1$ for $0 \leq y \leq \pi$. That solution will be called $U(x, y)$ in this problem.

a) Suppose $V(x, y)$ is the solution to $\Delta u = 0$ on S satisfying:
(BC) $u(x, 0) = 0$ & $u(x, \pi) = 1$ for $0 \leq x \leq \pi$; $u(0, y) = 0$ & $u(\pi, y) = 0$ for $0 \leq y \leq \pi$. Describe $V(x, y)$ in terms of $U(x, y)$.

b) Suppose $W(x, y)$ is the solution to $\Delta u = 0$ on S satisfying:
(BC) $u(x, 0) = 1$ & $u(x, \pi) = 0$ for $0 \leq x \leq \pi$; $u(0, y) = 0$ & $u(\pi, y) = 0$ for $0 \leq y \leq \pi$. Describe $W(x, y)$ in terms of $U(x, y)$.

c) Suppose $Z(x, y)$ is the solution to $\Delta u = 0$ on S satisfying:
(BC) $u(x, 0) = 5$ & $u(x, \pi) = -7$ for $0 \leq x \leq \pi$; $u(0, y) = 22$ & $u(\pi, y) = 4$ for $0 \leq y \leq \pi$. Describe $Z(x, y)$ in terms of $U(x, y)$.

Note *No significant computation* is needed in this problem: use linearity and ingenuity.

2. Find a solution to $\Delta u = 0$ on S satisfying
(BC) $u(x, 0) = 0$ & $u(x, \pi) = 0$ for $0 \leq x \leq \pi$; $u(0, y) = 0$ for $0 \leq y \leq \pi$; $u(\pi, y) = 0$ for $0 \leq y \leq \frac{\pi}{2}$ and $u(\pi, y) = 1$ for $\frac{\pi}{2} < y \leq \pi$.
The solution should be written in terms of an appropriate infinite series.

3. Suppose $F(x, y) = 7 \sin 3x \sin 8y - 9 \sin 11x \sin 4y$.

a) Find a solution to the heat equation $u_t = \Delta u$ in S subject to the following boundary and initial conditions:

(BC) $u(x, 0, 0) = 0$ & $u(x, \pi, 0) = 0$ for $0 \leq x \leq \pi$; $u(0, y, 0) = 0$ & $u(\pi, y, 0) = 0$ for $0 \leq y \leq \pi$

(IC) $u(x, y, 0) = F(x, y)$.

What happens to $u(x, y, t)$ as $t \rightarrow \infty$?

b) Find a solution to the wave equation $u_{tt} = \Delta u$ in S subject to the following boundary and initial conditions:

(BC) $u(x, 0, 0) = 0$ & $u(x, \pi, 0) = 0$ for $0 \leq x \leq \pi$; $u(0, y, 0) = 0$ & $u(\pi, y, 0) = 0$ for $0 \leq y \leq \pi$

(IC) $u(x, y, 0) = F(x, y)$ and $u_t(x, y, 0) = 0$.

What happens to $u(x, y, t)$ as $t \rightarrow \infty$?

4. Suppose $f(x, y)$ is 1 exactly when $0 \leq x \leq \frac{\pi}{2}$ and $0 \leq y \leq \frac{\pi}{3}$. Write $f(x, y)$ as the sum of a double sine series in S .

Note Here's a picture of a partial sum when the $\sin(nx) \sin(my)$'s satisfy $n, m \leq 50$.

