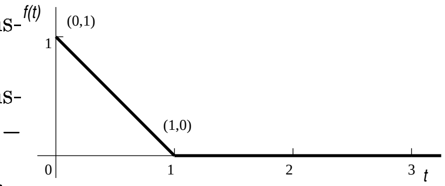
640:421:03 **Answers to the First Exam** **10/12/2004**

(16) 1. Here is a graph of the function $f(t)$ which is piecewise linear.



a) Use the definition of the Laplace transform to find the Laplace transform of the function $f(t)$.

Answer $f(t)$ is $1-t$ for $0 \le t < 1$ and 0 otherwise. The Laplace transform is $\mathcal{L}(f(t))(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 e^{-st}(1-t)\,dt = \int_0^1 e^{-st}\,dt - \int_0^1 e^{-st} t\,dt$. The first integral is: $\int_0^1 e^{-st}\,dt = -\left.\frac{e^{-st}}{s}\right]_{t=0}^{t=1} = -\frac{e^{-s}}{s} + \frac{1}{s}$.
Integrate by parts in the second: $\left.\begin{matrix} u = t \\ dv = e^{-st}dt \end{matrix}\right\}\left\{\begin{matrix} du = dt \\ v = -\frac{1}{s}e^{-st} \end{matrix}\right.$. Then $\int_0^1 e^{-st} t\,dt = t\left(-\frac{1}{s}e^{-st}\right)\Big]_{t=0}^{t=1} - \int_0^1 -\frac{1}{s}e^{-st}\,dt = -\frac{e^{-s}}{s} + \left(-\frac{1}{s^2}e^{-st}\right)\Big]_{t=0}^{t=1} = -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}$. There are three minus signs in $-\frac{e^{-s}}{s^2}$, two from antidifferentiation and one from integration by parts. The Laplace transform is $-\frac{e^{-s}}{s} + \frac{1}{s} - \left(-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right) = \frac{e^{-s}+s-1}{s^2}$ after combining terms, and noting that the second integral is subtracted.

b) Certainly $\int_0^\infty f(t)\,dt = \frac{1}{2}$. Use l'Hopital's rule to verify that $\lim_{s\to 0^+} F(s) = \frac{1}{2}$. Be sure to indicate why l'Hopital's rule applies each time you use it.

Answer $F(s) = \frac{e^{-s}+s-1}{s^2}$ When $s=0$, this is $\frac{0}{0}$. So (l'H) the limit is the same as the limit as $s \to 0^+$ of $\frac{-e^{-s}+1}{2s}$. Again, when $s=0$, this is $\frac{0}{0}$. l'H says consider the limit as $s \to 0^+$ of $\frac{e^{-s}}{2}$. This is $\frac{1}{2}$.

(14) 2. a) Use Laplace transforms to solve the initial value problem $y'' - y = 2$ with $\begin{cases} y(0) = 3 \\ y'(0) = 4 \end{cases}$.

Answer The Laplace transform of y'' is $s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 3s - 4$. The equation becomes $s^2Y(s) - 3s - 4 - Y(s) = \frac{2}{s}$ which then becomes $Y(s) = \frac{\frac{2}{s}+3s+4}{s^2-1}$. The rational function on the right-hand side is $\frac{3s^2-4s+2}{(s-1)(s+1)s}$. We use partial fractions: $\frac{3s^2+4s+2}{(s+1)(s-1)s} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1} = \frac{A(s+1)(s-1)+Bs(s-1)+Cs(s+1)}{(s-1)(s+1)s}$. Therefore we need A and B and C so that $3s^2+4s+2 = A(s+1)(s-1) + Bs(s-1) + Cs(s+1)$. If $s=0$ then $A=-2$. If $s=1$ then $C = \frac{9}{2}$. If $s=-1$ then $B = \frac{1}{2}$. Now we should find the inverse Laplace transform of $\frac{-2}{s} + \frac{\frac{1}{2}}{s+1} + \frac{\frac{9}{2}}{s-1}$. We can use the table to read off the answer: $y(t) = -2 + \frac{1}{2}e^{-t} + \frac{9}{2}e^t$.

b) Check that your answer satisfies the initial conditions.

Answer $y(0) = -2 + \frac{1}{2} + \frac{9}{2} = 5 - 2 = 3$. $y'(t) = \frac{9}{2}e^t - \frac{1}{2}e^{-t}$ so $y'(0) = \frac{9}{2} - \frac{1}{2}e^0 = 4$.

(12) 3. Find the Laplace transform of $\mathcal{U}(t-2)\left(3t^2 - e^{5t} + 2\right)$.

Answer This is of the form $\mathcal{U}(t-2)g(t)$, and its Laplace transform will look like $e^{-as}\mathcal{L}\{g(t+a)\}$ where $a = 2$ and $g(t) = 3t^2 - e^{5t} + 2$. Now $g(t+2) = 3(t+2)^2 - e^{5(t+2)} + 2 = 3t^2 + 12t + 12 - e^{10}e^{5t} + 2$, whose Laplace transform is $\frac{6}{s^3} + \frac{12}{s^2} + \frac{12}{s} - \frac{e^{10}}{s-5} + \frac{2}{s}$. The answer is therefore $e^{-2s}\left(\frac{6}{s^3} + \frac{12}{s^2} + \frac{12}{s} - \frac{e^{10}}{s-5} + \frac{2}{s}\right)$.

(12) 4. Compute the convolution of $\cos t$ and e^{2t}.

Answer The product of the Laplace transforms is the Laplace transform of the convolution, so the answer is the inverse Laplace transform of $\frac{s}{(s^2+1)(s-2)}$. Partial fractions again: $\frac{s}{(s^2+1)(s-2)} = \frac{As+B}{s^2+1} + \frac{C}{s-2} = \frac{(As+B)(s-2)+C(s^2+1)}{(s^2+1)(s-2)}$. So now we need $(As+B)(s-2) + C(s^2+1) = s$. $s=2$ gives $C = \frac{2}{5}$. Considering the s^2 coefficients gives $A + C = 0$ so $A = -\frac{2}{5}$. The constant terms on both sides gives $-2B + C = 0$ so $B = \frac{1}{5}$. The inverse Laplace transform of $-\frac{\frac{2}{5}s}{s^2+1} + \frac{\frac{1}{5}}{s^2+1} + \frac{\frac{2}{5}}{s-2}$ is $-\frac{2}{5}\cos t + \frac{1}{5}\sin t + \frac{2}{5}e^{2t}$.
We can also try to compute this directly as $\int_0^t \cos(t-\tau)e^{2\tau}\,d\tau$ but I would rather not, since the computation does involve two integrations by parts. I did check this direct computation with `Maple`, however, and got an answer agreeing with the computation above.

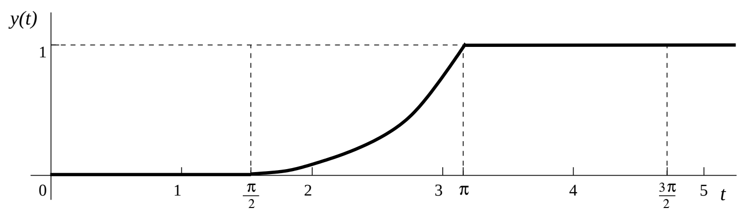
(20) 5. a) Solve the initial value problem $y'' + y = \mathcal{U}(t-\frac{\pi}{2}) - \delta(t-\pi)$ with $\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$.

Answer Laplace transform gives $s^2Y(s) + Y(s) = \frac{e^{\frac{\pi}{2}s}}{s} - e^{\pi s}$. Therefore $Y(s) = \frac{e^{-\frac{\pi}{2}s}}{(s^2+1)s} - \frac{e^{-\pi s}}{s^2+1}$. The inverse Laplace transform of the second term can be read off the table: $-\mathcal{U}(t-\pi)\sin(t-\pi)$. The other term is an exponential multiplying $\frac{1}{(s^2+1)s}$. That needs splitting up by partial fractions: $\frac{As+B}{s^2+1} + \frac{C}{s}$. Then $s(As+B) + C(s^2+1) = 1$ so $(s=0)$ $C=1$ and $B=0$ (s coefficient) and $A=-1$ (s^2 coefficient). Thus $\frac{1}{(s^2+1)s} = \frac{-s}{s^2+1} + \frac{1}{s}$ which maybe we could have guessed. We still need the inverse Laplace transform of $e^{-\frac{\pi}{2}s}\left(\frac{-s}{s^2+1} + \frac{1}{s}\right)$. We can read this off the table: $\mathcal{U}(t-\frac{\pi}{2})\left(-\cos(t-\frac{\pi}{2})+1\right)$. The complete answer is $y(t) = -\mathcal{U}(t-\pi)\sin(t-\pi) + \mathcal{U}(t-\frac{\pi}{2})\left(-\cos(t-\frac{\pi}{2})+1\right)$.

b) Write formulas *without* Heaviside functions for $y(t)$ in the indicated intervals:

Answer If $0 < t < \frac{\pi}{2}$ then $y(t) = 0$.
If $\frac{\pi}{2} < t < \pi$ then $y(t) = -\cos(t-\frac{\pi}{2}) + 1 = -\cos(t)\cos(-\frac{\pi}{2}) + \sin(t)\sin(-\frac{\pi}{2}) + 1 = 1 - \sin(t)$.
If $\pi < t$ then $y(t) = -\cos(t-\frac{\pi}{2}) + 1 - \sin(t-\pi) = 1 - \sin(t)$ (from the previous part) $-\big(\sin(t)\cos(-\pi) - \cos(t)\sin(-\pi)\big) = 1 - \sin(t) + \sin(t) = 1$. It seems a bit amazing that this is just 1 for all $t > \pi$.

c) Graph $y(t)$ as well as you can on the axes below.



d) For which t in the interval $0 < t < 5$ is $y(t)$ differentiable?

Answer Certainly $y(t)$ is differentiable away from $t = \frac{\pi}{2}$ and $t = \pi$. In fact, we are lucky because $1 - \sin(t)$ has a minimum at $\frac{\pi}{2}$, so that the function is also differentiable at $\frac{\pi}{2}$. The function is not differentiable at π.

(14) 6. Find a linear combination of $(t+1)^2$ and $(t+2)^2$ and $(t+3)^2$ which is equal to t^2.

Note You may use one of the RREF's supplied. If you do this, tell which one you use and describe how you use it.

Answer We need constants A and B and C so that $A(t+1)^2 + B(t+2)^2 + C(t+3)^2 = t^2$. We can "expand" the squares and get $A(t^2+2t+1) + B(t^2+4t+4) + C(t^2+6t+9) = t^2$. Then we can further get:

$$\begin{cases} 1A + 1B + 1C = 1 & \text{(from } t^2 \text{ coefficients)} \\ 2A + 4B + 6C = 0 & \text{(from } t \text{ coefficients)} \\ 1A + 4B + 9C = 0 & \text{(from constant coefficients)} \end{cases}$$

This system is an instantiation of the augmented matrix **PISCATAWAY** with $P = 1$ and $Q = 0$ and $R = 0$. The row-reduced form of **PISCATAWAY** allows me to conclude that $A = 3P - \frac{5}{4}Q + \frac{1}{2}R = 3$ and $B = -3P + 2Q - R = -3$ and $C = P + \frac{1}{2}Q - \frac{3}{4}R = 1$ answers the question: $t^2 = 3(t+1)^2 - 3(t+2)^2 + 1(t+3)^2$. You may check that this is correct.

Perhaps another way to solve this problem is "specializing", evaluating the candidate linear combination with specific t's. What we would get with three such choices is shown to the right. Then row reduction will give the same values of A and B and C.

$$\begin{cases} 1B + 4C = 1 & \text{(from } t = -1) \\ 1A + 1C = 4 & \text{(from } t = -2) \\ 4A + 1B = 9 & \text{(from } t = -3) \end{cases}$$

(12) 7. Prove that the three functions $\cos(t)$ and $\sin(t)$ and $\cos(2t)$ are linearly independent.

Answer We must show that if $A\cos(t) + B\sin(t) + C\cos(2t) = 0$ for all t, then $A = 0$ and $B = 0$ and $C = 0$. Here I will choose the special values $t = 0$ and $t = \frac{\pi}{2}$ and $t = \pi$. The following system is the result:

$\begin{cases} A + C = 0 \\ B - C = 0 \\ -A + C = 0 \end{cases}$. Now row-reduce: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and therefore the original system is equivalent to $A = 0$ and $B = 0$ and $C = 0$ which is what we were supposed to prove.