(12) 1. Complete the definitions. a) Suppose v_1, v_2, ... and v_t are vectors in $\mathbb{R}^n$. Then v_1, v_2, ... and v_t are *linearly independent* if *whenever* $\sum_{j=1}^t c_j v_j = 0$, *all of the scalars* c_j *must be 0.*
b) Suppose A is an n by n matrix. λ is an *eigenvalue* of A if *there is a non-zero vector* X *in* $\mathbb{R}^n$ *so that (writing* X *as a column vector)* $AX = \lambda X$ or if λ *is a root of* $\det(A - \lambda I_n) = 0$.

(22) 2. Suppose that $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. **Note** A is *not* symmetric! a) Compute the characteristic polynomial of A. **Answer** $\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^3 - (1-\lambda) = (1-\lambda)\lambda(-2+\lambda)$.

b) Find the eigenvalues of A. **Answer** 0 and 1 and 2.

c) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A. **Answer** Solve linear systems for each λ: $(A-\lambda)X = 0$.

0: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$; **1**: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$;

2: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ so $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

d) Find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP = D$. **Answer** $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$.

e) Find P^{-1}. **Answer**

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

so that $P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. f) Compute $Z = AP$. **Answer** $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

g) Compute $P^{-1}Z$ using the results of d) and e). **Answer** $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

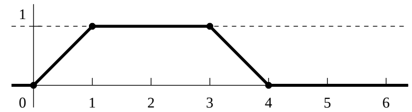
h) Write A as a product of D and P and P^{-1} (in the correct order!) and use this information to compute A^6. **Note** The entries in the answer are 0, 1, 31, or 32.
Answer Since $P^{-1}AP = D$, $A = PDP^{-1}$ so that $A^6 = PD^6P^{-1}$. And $D^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{pmatrix}$ so $PD^6 =$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 64 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 64 \\ 0 & -1 & 0 \\ 0 & 0 & 64 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 64 \\ 0 & -1 & 0 \\ 0 & 0 & 64 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 32 & 31 & 32 \\ 0 & 1 & 0 \\ 32 & 32 & 32 \end{pmatrix}$$

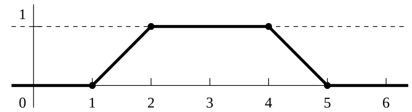
(12) 6. Suppose $f(x) = 3\sin(2x) - 5\cos(3x) + 2\cos(4x)$. Which of these integrals is larger: $\int_{-\pi}^{\pi} \left(f(x)\right)^2 dx$ or $\int_{-\pi}^{\pi} \left(f'(x)\right)^2 dx$? **Answer** Orthogonality implies $\int_{-\pi}^{\pi} \left(f(x)\right)^2 dx = \pi(9+25+4) = 38\pi$. $f'(x) = 6\sin(2x) - 15\cos(3x) + 8\cos(4x)$, so $\int_{-\pi}^{\pi} \left(f'(x)\right)^2 dx = \pi(36+225+64) = 325\pi$. The derivative integral is larger.

(8) 7. Suppose $f(x) = x + x^4$ for x in $[0,\pi]$.
a) If $F(x)$ is the odd extension of $f(x)$ to $[-\pi,\pi]$, *use the formula for* $f(x)$ *to write a simple formula or formulas for* $F(x)$. *Be sure you specify* $F(x)$ *for all* x*'s in* $[-\pi,\pi]$. **Answer** Since $F(-x) = -f(x)$, we see specifications can be $F(x) = x + x^4$ for x in $[0,\pi]$, and $F(x) = -\left((-x) + (-x)^4\right) = x - x^4$ for x in $[-\pi, 0)$. Which terms *must* be 0 in the Fourier series of $F(x)$? **Answer** All of the Fourier cosine terms (alternatively, the Fourier cosine coefficients must be 0).
b) If $G(x)$ is the even extension of $f(x)$ to $[-\pi,\pi]$, *use the formula for* $f(x)$ *to write a simple formula or formulas for* $G(x)$. *Be sure you specify* $G(x)$ *for all* x*'s in* $[-\pi,\pi]$. **Answer** Since $G(-x) = f(x)$, we see that specifications can be $G(x) = x + x^4$ for x in $[0,\pi]$, and $G(x) = (-x) + (-x)^4 = -x + x^4$ for x in $[-\pi, 0)$. Which terms *must* be 0 in the Fourier series of $G(x)$? **Answer** All of the Fourier sine terms (alternatively, the Fourier sine coefficients must be 0).

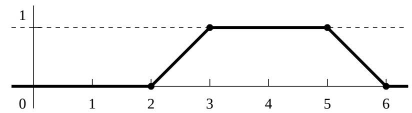
(16) 3. In this problem the functions $f(x)$ and $g(x)$ and $h(x)$ are piecewise linear functions. Parts of their graphs are shown to the right. The domains of these functions are all real numbers (all of $\mathbb{R}$). The functions are 0 where the graphs are not shown. a) Prove that the functions $f(x)$ and $g(x)$ and $h(x)$ are linearly independent.



The graph of $f(x)$



The graph of $g(x)$



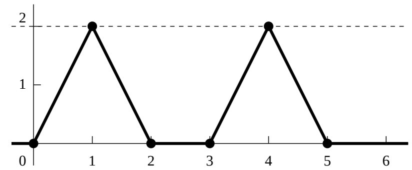
The graph of $h(x)$

Answer If $Af(x)+Bg(x)+Ch(x)=0$ then we can try some special values of x and get a system of linear equations:

$$\begin{cases} A=0 & \text{if } x=1; \\ A+B=0 & \text{if } x=2; \\ A+B+C=0 & \text{if } x=3. \end{cases}$$

$A=0$ from the first equation, the second shows that $B=0$, and the third shows that $C=0$. Therefore the only linear combination of $f(x)$ and $g(x)$ and $h(x)$ which is 0 is the trivial linear combination and the functions are linearly independent.

b) The function $Q(x)$ is piecewise linear and part of its graph is shown to the right. The domain of $Q(x)$ is all real numbers (all of $\mathbb{R}$) and the function $Q(x)$ is 0 where the graph is not shown. Can $Q(x)$ be written as a linear combination of $f(x)$ and $g(x)$ and $h(x)$?



The graph of $Q(x)$

Answer If $Q(x)=Af(x)+Bg(x)+Ch(x)$ then we can try some special values of x and get a system of linear equations:

$$\begin{cases} A=2 & \text{if } x=1; \\ A+B=0 & \text{if } x=2; \\ A+B+C=0 & \text{if } x=3; \\ B+C=2 & \text{if } x=4. \end{cases}$$

The first equation shows that A must be 2, then the second equation gives $B=-2$ and the third yields $C=0$. But the fourth equation gives $B=2$ which is a contradiction. There is no solution. $Q(x)$ cannot be written as a linear combination of $f(x)$, $g(x)$, and $h(x)$.

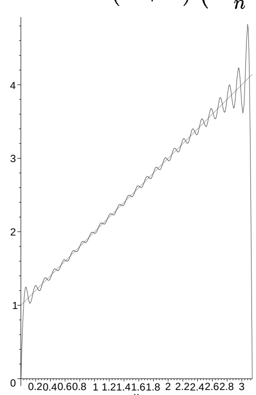
(12) 4. Suppose M is the matrix $\begin{pmatrix} a+b & 0 & 0 & 1 \\ b & -1 & 0 & 1 \\ c & 2 & -1 & 0 \\ b-c & 1 & -1 & 0 \end{pmatrix}$. Prove that M is <u>not invertible</u> exactly when the vector (a,b,c) in $\mathbb{R}^3$ is perpendicular to the vector $(1,1,-2)$ in $\mathbb{R}^3$. **Answer** We first compute $\det M$ by expanding along the first row. So $\det M = (a+b)\det\begin{pmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} - 1\det\begin{pmatrix} b & -1 & 0 \\ c & 2 & -1 \\ b-c & 1 & -1 \end{pmatrix}$. This is $(a+b)(-2+1)-1\Big(b(-2+1)-(-1)\big(-c+(b-c)\big)\Big) = -(a+b)-(-b-c+b-c) = -a-b+2c$. M is not invertible when this is 0, which is the same as requiring that the dot product of (a,b,c) and $(1,1,-2)$ is 0.

(18) 5. In this problem, $f(x)=x+1$. a) Compute $\int f(x)\sin(nx)\,dx$. **Answer** Integrate by parts:
$\left.\begin{matrix} u=x+1 \\ dv=\sin(nx)\,dx \end{matrix}\right\}\left\{\begin{matrix} du=dx \\ v=-\frac{1}{n}\cos(nx) \end{matrix}\right.$. The integral we want is $(x+1)\left(-\frac{1}{n}\cos(nx)\right)-\int -\frac{1}{n}\cos(nx)\,dx = (x+1)\left(-\frac{1}{n}\cos(nx)\right)+\frac{1}{n^2}\sin(nx)+C$.

b) Compute $b_n=\int_0^\pi f(x)\sin(nx)\,dx$ as explicitly as you can when n is a positive integer.
Answer $(x+1)\left(-\frac{1}{n}\cos(nx)\right)+\frac{1}{n^2}\sin(nx)\Big]_{x=0}^{x=\pi} = (\pi+1)\left(-\frac{1}{n}\right)(-1)^n-\left(-\frac{1}{n}\right)=(-1)^{n+1}\left(\frac{\pi+1}{n}\right)+\frac{1}{n}$

c) Give exact values for b_1 and b_2 and b_3 and b_4.
Answer $b_1=\pi+2$; $b_2=-\frac{1}{2}\pi$; $b_3=\frac{1}{3}\pi+\frac{2}{3}$; $b_4=-\frac{1}{4}\pi$.

d) Suppose $g(x)=\frac{2}{\pi}\sum_{n=1}^{100} b_n\sin(nx)$ and $h(x)=\frac{2}{\pi}\sum_{n=1}^{\infty} b_n\sin(nx)$. [*Below*] are two graphs of $f(x)=x+1$ for x in $[0,\pi]$. Sketch a reasonable approximation to $g(x)$ on the left graph. Sketch a reasonable approximation to $h(x)$ on the right graph.



⟸ Graph of $g(x)$, the 100[th] partial sum of the Fourier sine series on $[0,\pi]$ | Graph of $h(x)$, the sum of the whole Fourier sine series on $[0,\pi]$ ⟹