(10) 1. Find the Laplace transform of $\mathcal{U}(t-2)\left(t^3+e^{5t}\right)$.

(22) 2. a) Use Laplace transforms to solve the initial value problem $y''-2y'=3+\delta(t-5)$ with $\begin{cases} y(0)=1 \\ y'(0)=-1 \end{cases}$.

b) Write formulas *without* Heaviside functions for $y(t)$ in the indicated intervals:

If $0<t<5$ then $y(t)=$ ________________________ .

If $5<t$ then $y(t)=$ ________________________ .

c) Check that your answer satisfies the initial conditions.

$y(0)=$ ______ . For t near 0, $y'(t)=$ __________________ so that $y'(0)=$ ______ .

d) Is your solution continuous at $t=5$? Explain your answer *briefly*.

(18) 3. a) Complete the definition:

Suppose $v_1, v_2, \ldots$ and v_t are vectors in $\mathbb{R}^n$. Then $v_1, v_2, \ldots$ and v_t are *linearly independent* if

b) Prove that the functions $A(x)=(x-1)(x-2)$ and $B(x)=(x-1)^2$ and $C(x)=(x-2)^2$ are linearly independent.

(20) 4. Suppose that the matrix $A=\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$. **Note** A is *not* symmetric!

a) Compute the characteristic polynomial of A.

b) Find the eigenvalues of A.

c) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

d) Find a diagonal matrix D and an invertible matrix P so that $P^{-1}AP=D$.

e) Find P^{-1}.

f) Compute $Z=AP$.

g) Compute $P^{-1}Z$ using the results of e) and f).

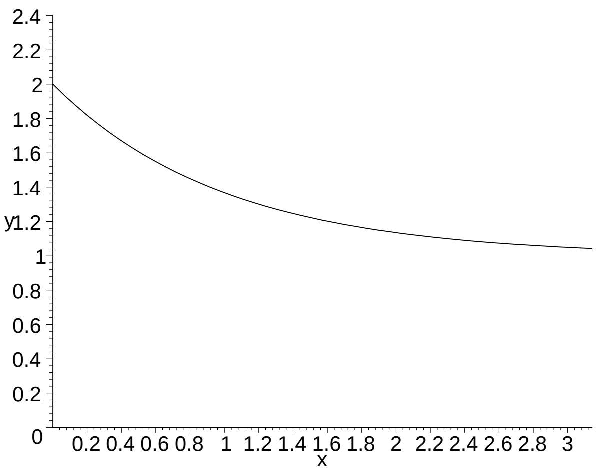
(20) 5. `Maple` states that $\int e^{Ax}\sin(Bx)\,dx = \frac{-Be^{Ax}\cos(Bx)}{A^2+B^2}+\frac{Ae^{Ax}\sin(Bx)}{A^2+B^2}$, a correct formula that you should use here. Suppose $f(x)=e^{-x}+1$.

a) Write the first four terms of the Fourier sine series for $f(x)$ on $[0,\pi]$ as simply as possible.

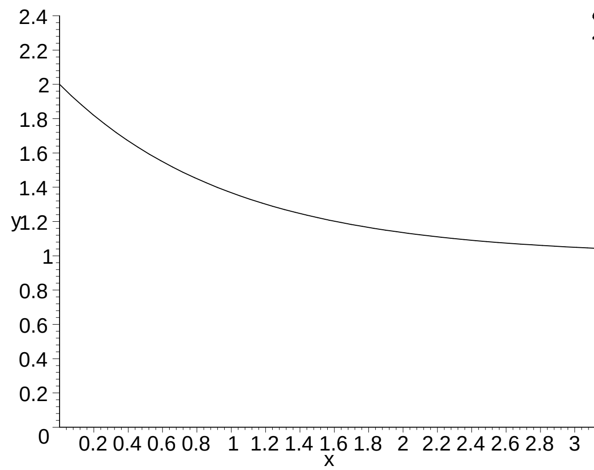
b) Suppose that $g(x)$ is the sum of the first 100 terms of the Fourier sine series for $f(x)$ on $[0,\pi]$, and $h(x)$ is the sum of the whole Fourier sine series for $f(x)$ on $[0,\pi]$.

Sketch a reasonable approximation to $g(x)$ on the left axes.

Sketch a reasonable approximation to $h(x)$ on the right axes. The graph of $y=f(x)$ is already drawn on each set of axes which should help.



Graph of $g(x)$, the 100^{th} partial sum of the Fourier sine series on $[0, \pi]$



Graph of $h(x)$, the sum of the whole Fourier sine series on $[0, \pi]$

(20) 6. Consider the following boundary value problem for $u(x, y)$, a function of two variables, with $0 \leq x \leq \pi$ and $y \geq 0$:

(PDE) $u_{xx} + u_y = 3u$.
(BC) $u(0, y) = 0$ and $u(\pi, y) = 0$ for all $y \geq 0$.

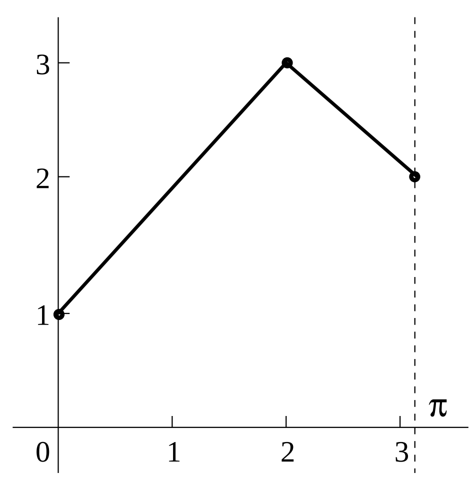
a) Use separation of variables to find product solutions $X(x)Y(y)$ to this boundary value problem. Be as explicit as you can about the resulting eigenvalues and eigenfunctions.

b) Now suppose the specifications of the boundary value problem include the following initial condition:

(IC) $u(x, 0) = \begin{cases} 1 & \text{if } \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \\ 0 & \text{all other } x\text{'s} \end{cases}$.

Use the product solutions found in a) together with the principle of superposition (linearity) to write a solution to the (PDE)+(BC)+(IC). Be as specific as you can about the resulting infinite series. In particular, write the first four terms of the series as explicitly as possible.

(12) 7. A bar of length π placed on the interval $[0, \pi]$ has one end, at $x = 0$, fixed at temperature 1 and the other end, at $x = \pi$, fixed at temperature 2 for all time. If $u(x, t)$ is the temperature in the bar at position x at time t, we suppose that the temperature satisfies the heat equation $u_{xx} = u_t$ (here $k = 1$) for all x in $[0, \pi]$ and $t \geq 0$. The graph of an initial temperature distribution $u(x, 0)$ is shown to the right. The graph shows a piecewise linear function connecting $(0, 1)$ and $(2, 3)$ and $(\pi, 2)$.
Suppose $u(x, t)$ is the solution to this boundary value problem.

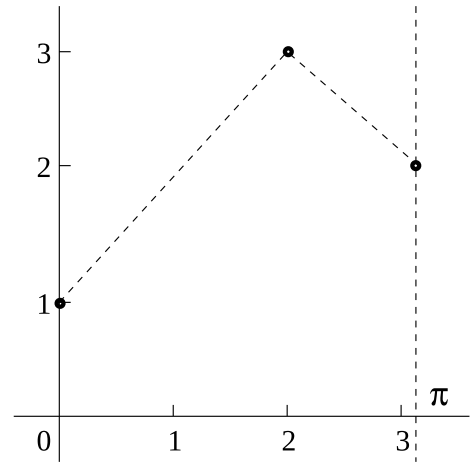


Initial temperature, $u(x, 0)$

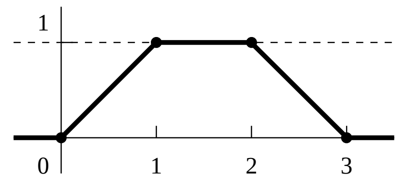
a) Sketch a graph of $u(x, \frac{1}{100})$ on the axes given below.

b) Sketch a graph of $u(x, 100)$ on the axes given below.

[The same picture, shown to the right, was given twice.]



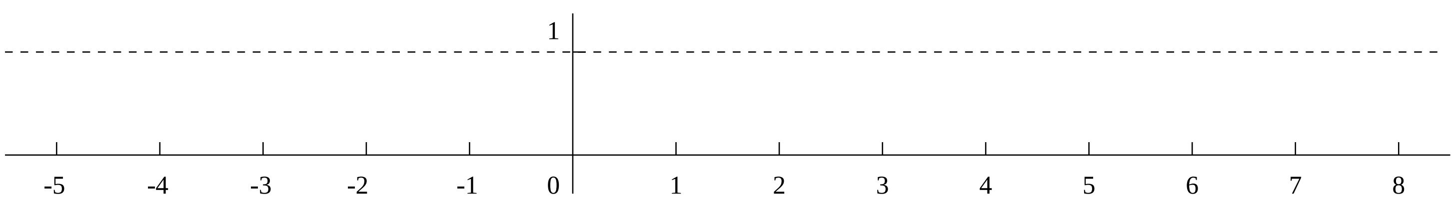
(16) 8. Consider the wave equation $u_{xx} = 4u_{tt}$ (here $c = 2$) for x in all of $\mathbb{R}$, the real numbers, with initial data $u(x, 0) = f(x)$ and $u_t(x, 0) = 0$. Here $f(x)$ is a piecewise linear function, a part of whose graph is shown to the right. $f(x)$ is 0 where the graph is not shown.



The graph of $f(x)$

a) Find $u(x, t)$. The formula in your answer will use the function $f(x)$.

b) Sketch $u(x, 2)$ on the axes given below.



c) At approximately what positive time will the displacement at $x = 30$ first be greater than 0?

d) Is $u(x, t)$ a differentiable (smooth) function of x and t? Explain your answer *briefly*.

(18) 9. Suppose $u(x, y, t)$ is a solution of the two-dimensional wave equation $u_{tt} = u_{xx} + u_{yy}$ for $0 \le x \le \pi$ and $0 \le y \le \pi$ and $t \ge 0$. Also suppose that $u(x, y, t)$ satisfies the following boundary and initial conditions:

(BC) $u(0, y, t) = 0$ and $u(\pi, y, t) = 0$ for all y with $0 \le y \le \pi$ and for all $t \ge 0$; $u(x, 0, t) = 0$ and $u(x, \pi, t) = 0$ for all x with $0 \le x \le \pi$ and for all $t \ge 0$;
(IC) $u(x, y, 0) = 6\sin(3x)\sin(4y) - 9\sin(5x)\sin(12y)$ and $u_t(x, y, 0) = 0$ for all x and y with $0 \le x \le \pi$ and $0 \le y \le \pi$.

a) Write a formula for $u(x, y, t)$.

b) How are $u(x, y, 0)$, the initial displacement, and $u(x, y, 2\pi)$, the displacement at time $t = 2\pi$, related? Explain your answer *briefly*.

c) Certainly $|u(x,y,t)| \leq 10^{10}$ for all x and y in $[0,\pi]$ and all $t \geq 0$. Give a much lower overestimate for the largest possible displacement at any time. Be sure to give evidence supporting your assertion.

d) Compute $E = \int_0^\pi \int_0^\pi \big(u(x,y,t)\big)^2\,dx\,dy$. *Use orthogonality appropriately* in this computation. Your answer, E, will be a function of time, t. What is the maximum value of E and at what t is this value attained?

(18) 10. Find the full Fourier series as explicitly as you can for the function $|x|$ (absolute value) in the interval $[-\pi,\pi]$. What does Parseval's formula state for this function and this series?

Final Exam for Math 421, section 3

December 21, 2004

NAME ______________________________

Do all problems, in any order.
Show your work. An answer alone may not receive full credit.
No notes other than the distributed formula sheet may be used on this exam.
No calculators may be used on this exam.

Problem Number	Possible Points	Points Earned:
1	10	
2	22	
3	18	
4	20	
5	20	
6	20	
7	12	
8	16	
9	18	
10	18	
Total Points Earned:		

Yes, the total number of points is 174. Please do the exam.