Math 421
Some Fourier series examples

Suppose we have told `Maple` about a function, $F(x)$. Below are some the `Maple` commands. The responses have generally not been given (they are mostly echos of the input).

These `Maple` commands compute the Fourier coefficients of $f(x)$:

```
>h:=n->(1/Pi)*int(F(x)*cos(n*x),x=0..2*Pi);
>g:=n->(1/Pi)*int(F(x)*sin(n*x),x=0..2*Pi);
```

This command computes a partial sum of the Fourier series of $F(x)$:

```
>Q:=N->h(0)/2+sum(h(n)*cos(n*x)+g(n)*sin(n*x),n=1..N);
```

Well, let's try all this. Here is an $F(x)$:

```
>F:=x->(1/10)*x^2;
```

I put the `1/10` in front so that certain pictures would come out better later. Can `Maple` integrate by parts?

```
>int(x^2*cos(x),x);
      2
     x  sin(x) - 2 sin(x) + 2 x cos(x)
```

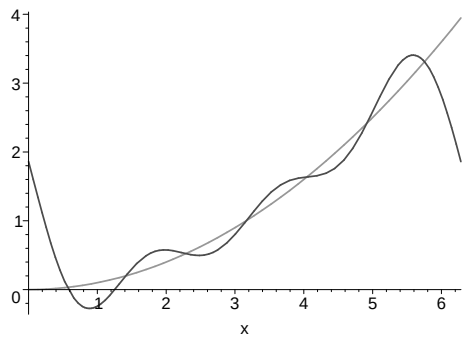
Now let's see what a partial sum of the Fourier series looks like::

```
>Q(3);
     2
    2 Pi
    ----- + 2/5 cos(x) - 2/5 Pi sin(x) + 1/10 cos(2 x)
     15
       - 1/5 Pi sin(2 x) + 2/45 cos(3 x) - 2/15 Pi sin(3 x)
```

O.k.: That's nice. But what can we *see* about the relationships between the original function and the partial sums? The command

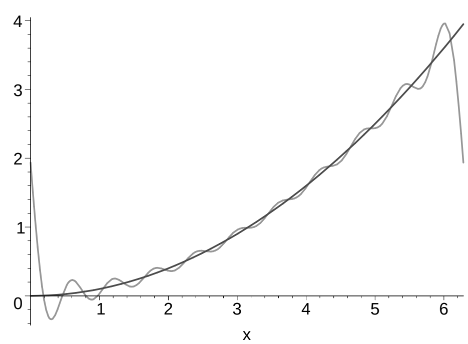
```
>plot({F(x),Q(3)},x=0..2*Pi,thickness=3,scaling=constrained);
```

will help. This command plots the original function and the third partial sum of the Fourier series of that function on the interval $[0, 2\pi]$. The thickness command draws the graphs a bit heavier or thicker than usual, and the scaling command asks that the horizontal and vertical proportions of the graph are scaled the same. The result is what is shown to the right.

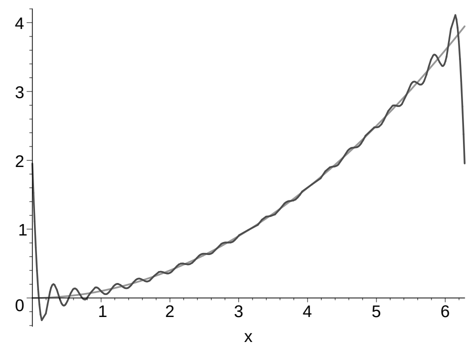


There's more on the other side.

The 10^{th} partial sum and $F(x)$



The 20^{th} partial sum and $F(x)$



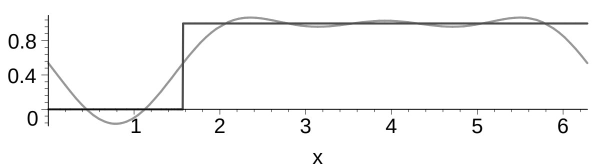
Here is something a bit stranger or maybe just more interesting: a piecewise-defined function. This is, of course, the function we met in Laplace transforms called $\mathcal{U}\left(t-\frac{\pi}{2}\right)$.

```
>F:=x->piecewise(x<Pi/2,0,1);
```

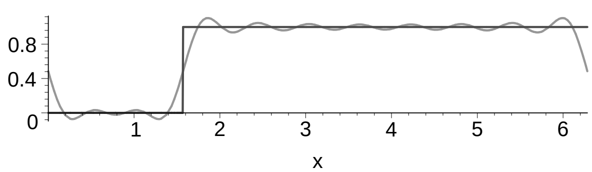
Just so you can see it, here is the partial sum up to the $n = 3$ terms of the Fourier series:

```
>Q(3);
              cos(x)   sin(x)   sin(2 x)       cos(3 x)       sin(3 x)
      3/4 - ------ - ------ - -------- + 1/3 -------- - 1/3 --------
                Pi       Pi        Pi             Pi             Pi
```

And now pictures, showing first the function and the 3^{rd} partial sum:



the function and the 10^{th} partial sum:



and finally, the function and the 20^{th} partial sum: