Math 421 **Some two-dimensional problems** **December 5, 2005**

The last two lectures will discuss some aspects of solutions of the heat and wave equations for two-dimensional regions. Sections 13.5 and 13.8 of the text contain some relevant material. In these problems, S denotes the π-by-π square with lower left corner at $(0,0)$.

1. In the lecture a solution to $\Delta u = u_{xx} + u_{yy} = 0$ on S was found which satisfied
(BC) $u(x,0) = 0$ & $u(x,\pi) = 0$ for $0 \le x \le \pi$; $u(0,y) = 0$ & $u(\pi,y) = 1$ for $0 \le y \le \pi$
That solution will be called $U(x,y)$ in this problem.

a) Suppose $V(x,y)$ is the solution to $\Delta u = 0$ on S satisfying:
(BC) $u(x,0) = 0$ & $u(x,\pi) = 1$ for $0 \le x \le \pi$; $u(0,y) = 0$ & $u(\pi,y) = 0$ for $0 \le y \le \pi$.
Describe $V(x,y)$ in terms of $U(x,y)$.

b) Suppose $W(x,y)$ is the solution to $\Delta u = 0$ on S satisfying:
(BC) $u(x,0) = 1$ & $u(x,\pi) = 0$ for $0 \le x \le \pi$; $u(0,y) = 0$ & $u(\pi,y) = 0$ for $0 \le y \le \pi$.
Describe $W(x,y)$ in terms of $U(x,y)$.

c) Suppose $Z(x,y)$ is the solution to $\Delta u = 0$ on S satisfying:
(BC) $u(x,0) = 5$ & $u(x,\pi) = -7$ for $0 \le x \le \pi$; $u(0,y) = 22$ & $u(\pi,y) = 4$ for $0 \le y \le \pi$.
Describe $Z(x,y)$ in terms of $U(x,y)$.

Note *No significant computation* is needed in this problem: use linearity and ingenuity.

2. Find a solution to $\Delta u = 0$ on S satisfying
(BC) $u(x,0) = 0$ & $u(x,\pi) = 0$ for $0 \le x \le \pi$; $u(0,y) = 0$ for $0 \le y \le \pi$; $u(\pi,y) = 0$ for $0 \le y \le \frac{\pi}{2}$ and $u(\pi,y) = 1$ for $\frac{\pi}{2} < y \le \pi$.
The solution should be written in terms of an appropriate infinite series.

3. Suppose $F(x,y) = 7\sin 3x \sin 8y - 9 \sin 11x \sin 4y$.

a) Find a solution to the heat equation $u_t = \Delta u$ in S subject to the following boundary and initial conditions:
(BC) $u(x,0,0) = 0$ & $u(x,\pi,0) = 0$ for $0 \le x \le \pi$; $u(0,y,0) = 0$ & $u(\pi,y,0) = 0$ for $0 \le y \le \pi$
(IC) $u(x,y,0) = F(x,y)$.
What happens to $u(x,y,t)$ as $\to \infty$?

b) Find a solution to the wave equation $u_{tt} = \Delta u$ in S subject to the following boundary and initial conditions:
(BC) $u(x,0,0) = 0$ & $u(x,\pi,0) = 0$ for $0 \le x \le \pi$; $u(0,y,0) = 0$ & $u(\pi,y,0) = 0$ for $0 \le y \le \pi$
(IC) $u(x,y,0) = F(x,y)$ and $u_t(x,y,0) = 0$.
What happens to $u(x,y,t)$ as $\to \infty$?

4. Suppose $f(x,y)$ is 1 exactly when $0 \le x \le \frac{\pi}{2}$ <u>and</u> $0 \le y \le \frac{\pi}{3}$. Write $f(x,y)$ as the sum of a double sine series in S.

Note Here's a picture of a partial sum when the $\sin(nx)\sin(my)$'s satisfy $n, m \le 50$.