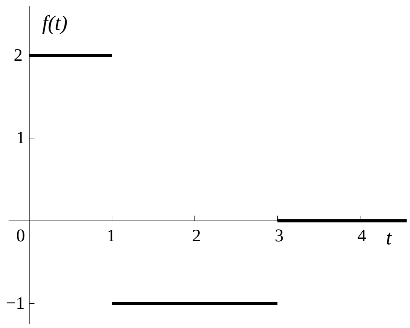
(16) 1. Here is a graph of the function $f(t)$ which is piecewise constant. The values of $f(t)$ are 2 and -1 and 0. $f(t)$ is 0 for all $t > 3$.



a) Use the definition of Laplace transform to find the Laplace transform $F(s)$ of the function $f(t)$.

Answer $F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^1 2e^{-st}\,dt + \int_1^3 -e^{-st}\,dt$. This is $\left.\frac{2e^{-st}}{-s}\right]_0^1 - \left.\frac{e^{-st}}{-s}\right]_1^3 = \frac{2-e^{-s}-2e^{-s}+e^{-3s}}{s} = \frac{2-3e^{-s}+e^{-3s}}{s}$.

b) Certainly $\int_0^\infty f(t)\,dt = 0$. Use l'Hopital's rule to verify that $\lim_{s\to 0^+} F(s) = 0$. Be sure to indicate why l'Hopital's rule is valid when you use it.

Answer The quotient $\frac{2-3e^{-s}+e^{-3s}}{s}$ has the form $\frac{0}{0}$ when $s = 0$. We use l'Hopital's rule: $\lim_{s\to 0^+} \frac{2-3e^{-s}+e^{-3s}}{s} \stackrel{\text{L'H}}{=} \lim_{s\to 0^+} \frac{3e^{-s}-3e^{-3s}}{1} = \frac{0}{1} = 0$.

(18) 2. a) Use Laplace transforms to solve the integrodifferential equation $y'(t) + 2\int_0^t y(\tau)\,d\tau = 3t$ with initial condition $y(0) = 0$.

Answer The Laplace transform of the equation is $sY(s) - y(0) + \frac{2}{s}Y(s) = \frac{3}{s^2}$. Insert the initial condition and solve for $Y(s)$: $Y(s) = \frac{3}{s^3+2s}$. Now use partial fractions: $\frac{3}{s^3+2s} = \frac{3}{(s^2+2)s} = \frac{A}{s} + \frac{Bs+C}{s^2+2}$ and we get $3 = A(s^2+2) + (Bs+C)s$. If $s = 0$, then $A = \frac{3}{2}$. The s^2 coefficient leads to $B = -\frac{3}{2}$ and the s coefficient shows that $C = 0$. Therefore we need the inverse Laplace transform of $\frac{\frac{3}{2}}{s} + \frac{-\frac{3}{2}s}{s^2+2}$ which can be read off the table: $\frac{3}{2} - \frac{3}{2}\cos(\sqrt{2}t)$.

b) Check that your answer satisfies the original equation. **Answer** $y'(t) = \underline{\frac{3\sqrt{2}}{2}\sin(\sqrt{2}t)}$ and $\int_0^t y(\tau)\,d\tau = \underline{\frac{3}{2}t - \frac{3}{2\sqrt{2}}\sin(\sqrt{2}t)}$ so that $y'(t) + 2\int_0^t y(\tau)\,d\tau = \underline{\frac{3\sqrt{2}}{2}\sin(\sqrt{2}t) + 2\left(\frac{3}{2}t - \frac{3}{2\sqrt{2}}\sin(\sqrt{2}t)\right) = 3t}$ since $\frac{3\sqrt{2}}{2} - 2\frac{3}{2\sqrt{2}} = 0$.

(20) 3. a) Compute $\int_0^5 \left(e^{3t}\right)\left(\mathcal{U}(t-2) + \delta(t-4)\right)dt$.

Answer $\int_0^5 \left(e^{3t}\right)\left(\mathcal{U}(t-2)+\delta(t-4)\right)dt = \int_0^5 e^{3t}\mathcal{U}(t-2)\,dt + \int_0^5 e^{3t}\delta(t-4)\,dt = \int_2^5 e^{3t}\,dt + e^{12} = \left.\frac{1}{3}e^{3t}\right]_2^5 + e^{12} = \frac{1}{3}(e^{15} - e^6) + e^{12}$.

b) Compute the Laplace transform of $\mathcal{U}(t-3)\left(4t + e^{7t}\right)$.

Answer This is e^{-3s} multiplied by the Laplace transform of the result of substituting $t+3$ for t in $4t + e^{7t}$. That substitution gives $4(t+3) + e^{7(t+3)} = 4t + 12 + e^{21}e^{7t}$ which has Laplace transform $\frac{4}{s^2} + \frac{12}{s} + \frac{e^{21}}{s-7}$ so the answer is $e^{-3s}\left(\frac{4}{s^2} + \frac{12}{s} + \frac{e^{21}}{s-7}\right)$.

c) Compute the convolution of t and e^{2t}.

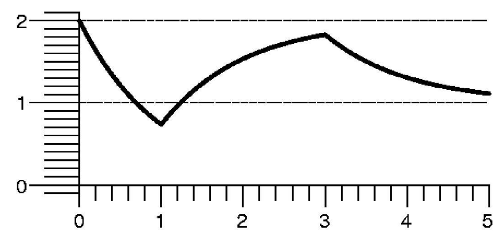
Answer The product of the Laplace transforms is the Laplace transform of the convolution, so the answer is the inverse Laplace transform of $\frac{1}{s^2(s-2)}$. Partial fractions again: $\frac{1}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2}$ resulting in $1 = As(s-2) + B(s-2) + Cs^2$. When $s = 0$, we get $B = -\frac{1}{2}$. When $s = 2$, $C = \frac{1}{4}$. The s^2 coefficient gives $A = -\frac{1}{4}$. The inverse Laplace transform of $\frac{-\frac{1}{4}}{s} + \frac{-\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s-2}$ is $-\frac{1}{4} - \frac{1}{2}t + \frac{1}{4}e^{2t}$.

The convolution requested can also be computed using the definition: $\int_0^t (t-\tau)e^{2\tau}\,d\tau = \left.\frac{t}{2}e^{2\tau} - \left(\frac{\tau e^{2\tau}}{2} - \frac{e^{2\tau}}{4}\right)\right]_0^t =$ the same answer. The antidifferentiation uses one integration by parts.

(20) 4. a) Solve the initial value problem $y' + y = 2\mathcal{U}(t-1) - \mathcal{U}(t-3)$ with $y(0) = 2$.

Answer The Laplace transform of the equation is $sY(s) - 2 + Y(s) = \frac{2e^{-s}}{s} - \frac{e^{-3s}}{s}$. We solve for $Y(s)$ and get $Y(s) = \frac{2e^{-s}-e^{-3s}+2s}{s(s+1)} = \frac{2e^{-s}-e^{-3s}}{s(s+1)} + \frac{2}{s+1}$. The inverse Laplace transform of the last piece is just $2e^{-t}$. For the rest, we use partial fractions on $\frac{1}{s(s+1)}$. The result is $\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$. Now consider $\left(2e^{-s} - e^{-3s}\right)\left(\frac{1}{s} - \frac{1}{s+1}\right)$. The inverse Laplace transform of $2e^{-s}\left(\frac{1}{s} - \frac{1}{s+1}\right)$ is $2\mathcal{U}(t-1)\left(1 - e^{-(t-1)}\right)$ and the inverse Laplace transform of $-e^{-3s}\left(\frac{1}{s} - \frac{1}{s+1}\right)$ is $-\mathcal{U}(t-3)\left(1 - e^{-(t-3)}\right)$.

And finally, $y(t) = 2\mathcal{U}(t-1)\left(1-e^{-(t-1)}\right) - \mathcal{U}(t-3)\left(1-e^{-(t-3)}\right) + 2e^{-t}$.
b) Write formulas *without* Heaviside functions for $y(t)$ in the indicated intervals:
Answer If $0 < t < 1$ then $y(t) = \underline{2e^{-t}}$.
If $1 < t < 3$ then $y(t) = \underline{2e^{-t} + 2 - 2e^{-(t-1)}}$.
If $3 < t$ then $y(t) = \underline{2e^{-t} + 2 - 2e^{-(t-1)} - 1 + e^{-(t-3)}}$.
c) Graph $y(t)$ as well as you can on the axes below.
Answer In the interval $1 < t < 3$, we can rewrite the formula for $y(t)$ as $2 + (2-2e)\,e^{-t}$, and see that as $t \to 1^+$, this $\to 2 + (2-2e)e^{-1} = 2e^{-1}$. This is the same as $\lim_{t\to 1^-} 2e^{-t}$. The exponential's coefficient, $2-2e$, is negative. Therefore the curve is concave down.
For $t > 3$, the formula for $y(t)$ becomes $1 + \left(2 - 2e + e^3\right) e^{-t}$. This part of the curve is concave up because $2 - 2e + e^3$ is positive. As $t \to 3^+$, this formula $\to 1 + \left(2 - 2e + e^3\right) e^{-3} = 1 + 2e^{-3} - 2e^{-2} + 1$ which is the same as $\lim_{t\to 3^-} 2 + (2-2e)\,e^{-t}$.



d) For which t in the interval $0 < t < 5$ is $y(t)$ continuous?
Answer All t.
e) For which t in the interval $0 < t < 5$ is $y(t)$ differentiable?
Answer All t except $t = 3$ and $= 1$.
f) What is $\lim_{t\to\infty} y(t)$?
Answer $1 + \left(2 - 2e + e^3\right) e^{-t} \to 1$ as $\to \infty$ since $e^{-t} \to 0$ as $\to \infty$.

(14) 5. Find a linear combination of $(2,1,-1,1)$ and $(-1,1,1,2)$ and $(1,1,3,-2)$ which is equal to $(7,1,-11,6)$.
Note You may use one of the RREF's supplied. If you do this, tell which one you use and describe how you use it.
Answer We need constants A and B and C so that $A(2,1,-1,1) + B(-1,1,1,2) + C(1,1,3,-2) = (7,1,-11,6)$. This is just

$$\begin{cases} 2A - 1B + 1C = 7 \\ 1A + 1B + 1C = 1 \\ -1A + 1B + 3C = -11 \\ 1A + 2B - 2C = 6 \end{cases}$$

This system is an instantiation of the augmented matrix **PACIFIC** with $P = 7$, $Q = 1$, $R = -11$, and $S = 6$. The row-reduced form of **PACIFIC** allows us to conclude that $A = \frac{1}{5}P + \frac{2}{5}Q - \frac{1}{5}R = \frac{7}{5} + \frac{2}{5} + \frac{11}{5} = 4$, $B = -\frac{2}{5}P + \frac{7}{10}Q - \frac{1}{10}R = -\frac{14}{5} + \frac{7}{10} + \frac{11}{10} = -1$, and $C = \frac{1}{5}P - \frac{1}{10}Q + \frac{3}{10}R = \frac{7}{5} - \frac{1}{10} - \frac{33}{10} = -2$. The last equation, $P - 2Q + R + S = 0$, furnishes a useful way to check since $7 - 2(1) - 11 + 6$ is equal to 0. Therefore $4(2,1,-1,1) - 1(-1,1,1,2) - 2(1,1,3,-2) = (7,1,-11,6)$.

(12) 6. Prove that the three functions t^3 and $t^2(t-1)$ and $t(t-1)(t+1)$ are linearly independent.
Answer Suppose that $At^3 + Bt^2(t-1) + Ct(t-1)(t+1) = 0$ for some constants, A, B, and C, and for all t. We must verify that all of these constants are 0. Only one expression has a t coefficient, the last. Thus $C = 0$. Then we consider $At^3 + Bt^2(t-1) = 0$. Only one expression has a t^2 coefficient, the last. Thus $B = 0$. Finally we have $At^3 = 0$ so that A must be 0.
Certainly there are other ways to do this problem successfully. For example, set $t = 1$, and then observe that A must be 0, etc.