640:421:01 **Answers to the Second Exam** **11/18/2004**

(12) 1. Complete the definitions.

a) Suppose A is a $p \times q$ matrix. The *transpose of* A, A^t, is *a* $q \times p$ *matrix whose* $(i,j)^{\text{th}}$ *entry is the* $(j,i)^{\text{th}}$ *entry of* A.

b) Suppose v_1, v_2, $\ldots$ and v_t are vectors in $\mathbb{R}^n$. Then v_1, v_2, $\ldots$ and v_t are *linearly independent* if *whenever* $\sum_{j=1}^t c_j v_j = 0$, *all of the scalars* c_j *must be* 0.

(20) 2. Suppose that $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$.

a) Compute the characteristic polynomial of A.

Answer $\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{pmatrix} = (1-\lambda)(-\lambda) - 2 \cdot 3(1-\lambda) = \lambda^2 - \lambda - 6$.

b) Find the eigenvalues of A.

Answer Since $\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$, the eigenvalues are 3 and -2.

c) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

Answer Solve the linear system $(A - \lambda)X = 0$ for each λ with $X \neq 0$.

For $\lambda = 3$: $\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$;

For $\lambda = -2$: $\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

d) Find a diagonal matrix D and an invertible matrix C so that $C^{-1}AC = D$.

Answer $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$.

e) Find C^{-1}.

Answer $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & -3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -5 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & \frac{2}{5} \\ 0 & 1 & \frac{1}{5} & -\frac{1}{5} \end{array}\right)$ so $C^{-1} = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{pmatrix}$.

f) Compute $Z = AC$.

Answer $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 3 & 6 \end{pmatrix}$

g) Compute $C^{-1}Z$ using the results of e) and f).

Answer $\begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 3 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$.

h) Write A as a product of D and C and C^{-1} (in the correct order!) and then use this information to compute $A - A^2 + A^3$.

Note The entries in the matrix which is the answer are 0, 7, 14, and 21.

Please answer the question asked; a direct computation without C will earn no points.

Answer Since $C^{-1}AC = D$, $A = CDC^{-1}$ so that $A - A^2 + A^3 = C\left(D - D^2 + D^3\right)C^{-1}$.

And $D - D^2 + D^3 = \begin{pmatrix} 3 - 9 + 27 & 0 \\ 0 & -2 - 4 - 8 \end{pmatrix} = \begin{pmatrix} 21 & 0 \\ 0 & -14 \end{pmatrix}$ so $C\left(D - D^2 + D^3\right) =$

$\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 21 & 0 \\ 0 & -14 \end{pmatrix} = \begin{pmatrix} 21 & -28 \\ 21 & 42 \end{pmatrix}$ and $\begin{pmatrix} 21 & -28 \\ 21 & 42 \end{pmatrix}\begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{1}{5} \end{pmatrix} = \begin{pmatrix} 7 & 14 \\ 21 & 0 \end{pmatrix}$.

(8) 3. Explain why the matrix $A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ *cannot* be diagonalized.

Answer $\det(A - \lambda I) = -\lambda^3$. The only eigenvalue is 0. The eigenvectors are obtained by solving the system $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ which is $\begin{cases} y + 2z = 0 \\ 3z = 0 \\ 0 = 0 \end{cases}$. The third equation has no information. The second equation implies that $z = 0$ and then the first equation implies that $y = 0$. The only eigenvectors are non-zero multiples of $(1, 0, 0)$. There are not enough linearly independent eigenvectors to diagonalize the matrix.

(12) 4. Let A be a non-zero 4×6 matrix.
a) If $\operatorname{rank}(A|B) = 2$ (here B is a 4×1 column matrix), then for what value(s) of $\operatorname{rank}(A)$ is the system $AX = B$, $B \neq 0$, inconsistent? Consistent? Briefly explain your answers.
Comment First state what the possible values of $\operatorname{rank}(A)$ are if $\operatorname{rank}(A|B) = 2$ and why.
Answer The rank can only *increase* with more columns, so the rank of A must be either 1 or 2. It cannot be 0 since we are told that A is non-zero. If $\operatorname{rank}(A) = 1$ then B supplies an extra non-zero row in the RREF of $(A|B)$. This means that the system of equations cannot be solved, and is therefore inconsistent. If $\operatorname{rank}(A) = 2$ there are no extra compatibility conditions, so the system of equations can be solved, and is therefore consistent.
b) If $\operatorname{rank}(A) = 3$, then how many parameters does the solution of the system $AX = 0$ have? Briefly explain your answer.
Answer When $\operatorname{rank}(A) = 3$, there are 3 more columns in the RREF of A which are in addition to the columns with leading 1's. These columns designate variables which are *free* and therefore there are 3 parameters to the solution of this homogeneous system.
Remark Students were asked to hand in solutions to problem 16 in section 8.3. This question is most of that problem.

(12) 5. Suppose M is the 5×5 matrix $\begin{pmatrix} a & 0 & 0 & a & a \\ 0 & 0 & b & 0 & 0 \\ c & c & 0 & c & 0 \\ d & 0 & d & d & 0 \\ 0 & e & 0 & e & e \end{pmatrix}$. The determinant of M is a scalar multiple of $abcde$. Compute $\det(M)$.
Reminder Show your work. An answer alone will not receive full credit. Be careful!
Answer There are many ways to do this. I'll expand along the last column. So $\det(M) =$ $+a\det\begin{pmatrix} 0 & 0 & b & 0 \\ c & c & 0 & c \\ d & 0 & d & d \\ 0 & e & 0 & e \end{pmatrix} + e\det\begin{pmatrix} a & 0 & 0 & a \\ 0 & 0 & b & 0 \\ c & c & 0 & c \\ d & 0 & d & d \end{pmatrix}$. For the first 4×4 determinant, I'll expand along the first row. So $\det\begin{pmatrix} 0 & 0 & b & 0 \\ c & c & 0 & c \\ d & 0 & d & d \\ 0 & e & 0 & e \end{pmatrix} = +b\det\begin{pmatrix} c & c & c \\ d & 0 & d \\ 0 & e & e \end{pmatrix} = b(cde - cde - cde) = -bcde$ since 3×3 determinants are easy, especially when there are lots of 0's.

I'll expand the other 4×4 determinant along the second column: $\det \begin{pmatrix} a & 0 & 0 & a \\ 0 & 0 & b & 0 \\ c & c & 0 & c \\ d & 0 & d & d \end{pmatrix} =$ $-c \det \begin{pmatrix} a & 0 & a \\ 0 & b & 0 \\ d & d & d \end{pmatrix} = -c(abd - abd) = 0$. Therefore the determinant of M is $-abcde$.

(10) 7. a) Suppose the function $A + Bx^2$ is orthogonal to both the function 1 and the function x on the interval $[0, 1]$. Prove that both A and B must be 0.

Answer $\int_0^1 1(A + Bx^2)\,dx = A + \frac{1}{3}B$ and $\int_0^1 x(A + Bx^2)\,dx = \frac{1}{2}A + \frac{1}{4}B$. $A + Bx^2$ is orthogonal to both 1 and x if $\begin{cases} A + \frac{1}{3}B = 0 \\ \frac{1}{2}A + \frac{1}{4}B = 0 \end{cases}$. Since $\det \begin{pmatrix} 1 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} = \frac{1}{4} - \frac{1}{6} \neq 0$, the system has only the trivial solution.

b) Find one example of a non-zero function of the form $A + Bx^2$ which is orthogonal to both the function 1 and the function x on the interval $[-1, 1]$.

Answer $\int_{-1}^1 1(A+Bx^2)\,dx = 2A + \frac{2}{3}B$ and $\int_{-1}^1 x(A+Bx^2)\,dx = 0$. $A+Bx^2$ is orthogonal to both 1 and x exactly when $2A + \frac{2}{3}B = 0$ since the other equation has no information. One solution is $A = -1$ and $B = 3$ so the function is $-1 + 3x^2$. A non-zero multiple of this function is also correct.

(18) 6. In this problem, $f(x)$ is a function whose domain is $[-\pi, \pi]$ and which is defined by the piecewise formula $f(x) = \begin{cases} 0 & \text{if } x < -\frac{\pi}{2} \\ x & \text{if } x \geq -\frac{\pi}{2} \end{cases}$. A graph of $f(x)$ is on the next page.

a) Compute $\int x \sin(nx)\,dx$ if n is not 0. **Answer** Use $\left. \begin{matrix} u = x \\ dv = \sin(nx)\,dx \end{matrix} \right\} \left\{ \begin{matrix} du = dx \\ v = -\frac{1}{n}\cos(nx) \end{matrix} \right.$ so the integral we want is $x\left(-\frac{1}{n}\cos(nx)\right) - \int -\frac{1}{n}\cos(nx)\,dx = -\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2} + C$.

b) Use the notation for the coefficients of the Fourier series of $f(x)$ on the formula sheet. Find the following Fourier coefficients. The answers for $n > 0$ are sums of rational numbers (quotients of integers) and rational multiples of $\frac{1}{\pi}$. Use the formula you got in a) and the following result: if $n \neq 0$, $\int x\cos(nx)\,dx = \frac{\cos(nx)}{n^2} + \frac{x\sin(nx)}{n} + C$.

Answer $\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\frac{\pi}{2}}^{\pi} x\,dx = \frac{1}{2}x^2\Big]_{-\frac{\pi}{2}}^{\pi} = \frac{1}{2}\left(\pi^2 - \frac{\pi^2}{4}\right) = \frac{3}{8}\pi^2$. Divide this by 2π to get the exact value of the constant term: $\underline{\frac{3\pi}{16}}$.

Since $\frac{1}{\pi}\left[\left(\frac{\cos(1\pi)}{1^2} + \frac{\pi\sin(1\pi)}{1}\right) - \left(\frac{\cos(1(-\frac{\pi}{2}))}{1^2} + \frac{-\frac{\pi}{2}\sin(1(-\frac{\pi}{2}))}{1}\right)\right] = \frac{1}{\pi}\left[-1 - \frac{\pi}{2}\right]$, the exact value of the first Fourier cosine coefficient, a_1, is $\underline{-\frac{1}{\pi} - \frac{1}{2}}$.

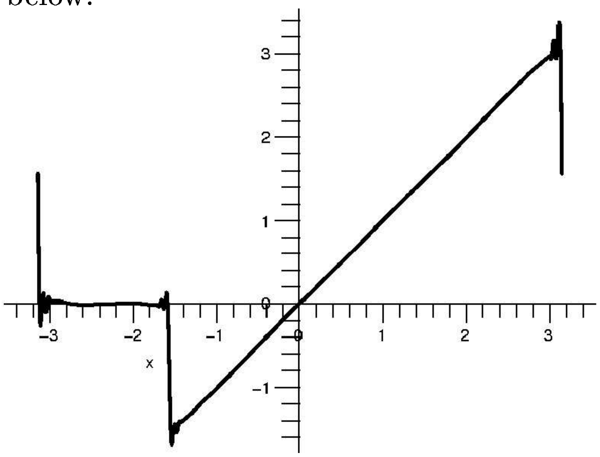
Since $\frac{1}{\pi}\left[\left(\frac{\cos(2\pi)}{2^2} + \frac{\pi\sin(2\pi)}{2}\right) - \left(\frac{\cos(2(-\frac{\pi}{2}))}{2^2} + \frac{-\frac{\pi}{2}\sin(2(-\frac{\pi}{2}))}{1}\right)\right] = \frac{1}{\pi}\left[\frac{1}{4} - -\frac{1}{4}\right]$, the exact value of the second Fourier cosine coefficient, a_2, is $\underline{\frac{1}{2\pi}}$.

Since $\frac{1}{\pi}\left[\left(-\frac{\pi\cos(1\pi)}{1} + \frac{\sin(1\pi)}{1^2}\right) - \left(-\frac{-\frac{\pi}{2}\cos(1(-\frac{\pi}{2}))}{1} + \frac{\sin(1(-\frac{\pi}{2}))}{1^2}\right)\right] = \frac{1}{\pi}\left[\pi + 1\right]$, the exact value of the first Fourier sine coefficient, b_1, is $\underline{1 + \frac{1}{\pi}}$.

Since $\frac{1}{\pi}\left[\left(-\frac{\pi\cos(2\pi)}{2} + \frac{\sin(2\pi)}{2^2}\right) - \left(-\frac{-\frac{\pi}{2}\cos(2(-\frac{\pi}{2}))}{2} + \frac{\sin(2(-\frac{\pi}{2}))}{2^2}\right)\right] = \frac{1}{\pi}\left[-\frac{\pi}{2} + \frac{\pi}{4}\right]$, the exact value of the second Fourier sine coefficient, b_2, is $\underline{-\frac{1}{4}}$.

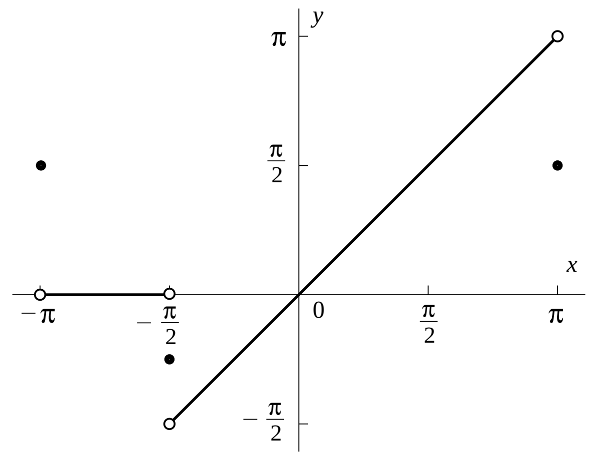
c) Suppose $g(x)$ is the partial sum up to the $n=100$ terms in both sine and cosine for the Fourier series of $f(x)$, and $h(x)$ is the sum of the full Fourier series of $f(x)$.
Below are two graphs of $f(x)$ for x in $[-\pi, \pi]$.

Sketch a reasonable approximation to the graph of $y = g(x)$ on the axes [to the right] below.



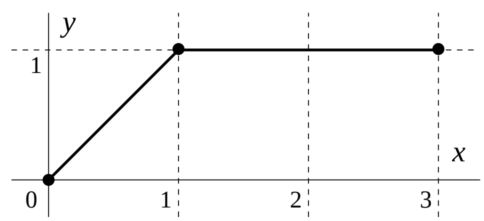
The student should sketch here the graph of $y = g(x)$, the 100^{th} partial sum of the Fourier series of $f(x)$ on $[-\pi, \pi]$.

Sketch a reasonable approximation to the graph of $y = h(x)$ on the axes [to the right] below.

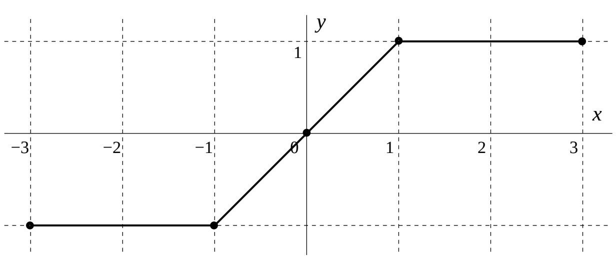


The student should sketch here the graph of $y = h(x)$, the sum of the whole Fourier series of $f(x)$ on $[-\pi, \pi]$.

(8) 8. A graph of $y = f(x)$ is shown to the right. $f(x)$ is a piecewise linear function and its domain is $[0, 3]$.

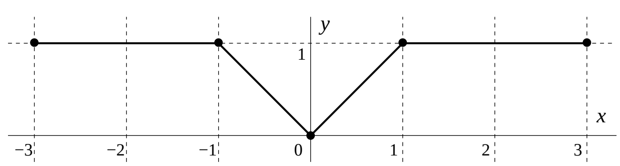


a) Suppose $F(x)$ is the odd extension of $f(x)$ to $[-3, 3]$. Graph $y = F(x)$ on the axes provided below.



Which terms *must* be 0 in the Fourier series of $F(x)$? (Here only an answer is requested, with no explanation needed.)
ANSWER: All of the cosine terms.

b) Suppose $G(x)$ is the even extension of $f(x)$ to $[-3, 3]$. Graph $y = G(x)$ on the axes provided below.



Which terms *must* be 0 in the Fourier series of $G(x)$? (Here only an answer is requested, with no explanation needed.)
ANSWER: All of the sine terms.