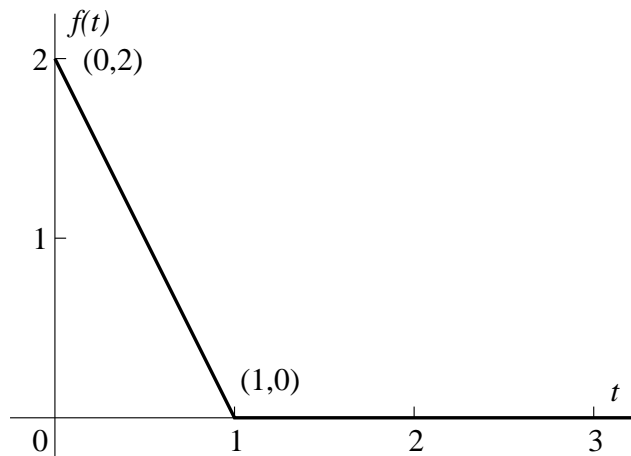(18) 1. Here is a graph of the piecewise linear function $f(t)$. $f(t)$ is defined for $t \geq 0$. The graph connects the points $(0, 2)$ and $(1, 0)$. $f(t) = 0$ for all $t \geq 1$.

a) <u>Use the definition of the Laplace transform</u> to find the Laplace transform $F(s)$ of the function $f(t)$.

b) Find $\lim\limits_{s \to 0^+} F(s)$. You will need to use l'Hopital's rule. Be sure to indicate why l'Hopital's rule applies each time you use it.

(18) 2. a) Use Laplace transforms to solve the initial value problem $y'' - y = \delta(t - 2)$ with
$\begin{cases} y(0) = 3 \\ y'(0) = 4 \end{cases}$.

b) Write formulas *without* Heaviside functions for the answer to a) in the intervals given.
If $0 < t < 2$ then $y(t) =$ _____ .
If $2 < t$ then $y(t) =$ _____ .
c) Check that the answer to a) satisfies the initial conditions.
$y(0) =$ _____ .
For $t < 2$, $y'(t) =$ _____ so that $y'(0) =$ _____ .

(12) 3. a) Complete the definition:

Suppose A is an n by n matrix. v, a vector in R^n, is an *eigenvector* of A if

b) Verify that $(1, -2, -2)$ is an eigenvector of $A = \begin{pmatrix} 29 & -9 & 24 \\ -54 & 20 & -48 \\ -54 & 18 & -46 \end{pmatrix}$. What is the

eigenvalue corresponding to this eigenvector?

(18) 4. a) Compute the characteristic polynomial of $A = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

b) What are the eigenvalues of A?
c) It is true and you do not need to verify (and should not!) that A has exactly 1 linearly independent eigenvector for each eigenvalue. Use your answer to b) to decide if A can be diagonalized. Explain why your answer is correct.
d) What is the rank of A? Why?

(12) 5. a) Prove that the functions $f(x) = x$ and $g(x) = x - 1$ and $h(x) = x(x - 1)$ are linearly independent.
b) Verify that $F(x) = x^3$ *cannot* be written as a linear combination of the functions $f(x)$ and $g(x)$ and $h(x)$ defined in a).

(16)   6. `Maple` declares: $\int x^2 \sin(nx)\, dx = \dfrac{-n^2 x^2 \cos(nx) + 2\cos(nx) + 2nx\sin(nx)}{n^3}$ for $n \neq 0$.

You should use this formula here. Suppose $f(x) = \begin{cases} x^2 & \text{if } 0 \leq x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \leq x \leq \pi \end{cases}$.

a) Write the first three terms (*not* just the coefficients!) of the Fourier sine series for $f(x)$ on $[0, \pi]$ as simply as possible.

b) Suppose that $g(x)$ is the sum of the first 100 terms of the Fourier sine series for $f(x)$ on $[0, \pi]$, and $h(x)$ is the sum of the whole Fourier sine series for $f(x)$ on $[0, \pi]$.
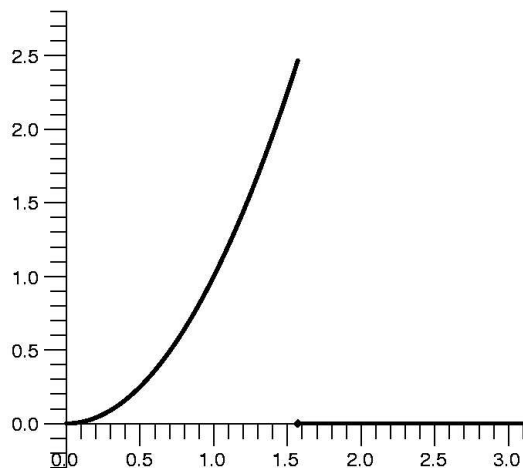Sketch a reasonable approximation to $g(x)$ on the left axes.
Sketch a reasonable approximation to $h(x)$ on the right axes.
A `Maple` graph of $y = f(x)$ is already drawn on each set of axes which should help.
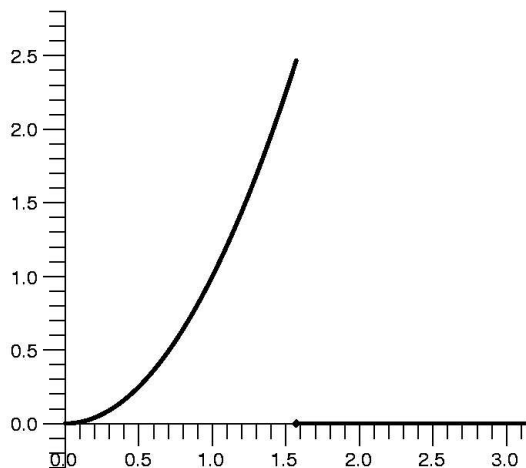
Graph of $g(x)$, the 100th partial sum
of the Fourier sine series on $[0, \pi]$

Graph of $h(x)$, the sum of the whole
Fourier sine series on $[0, \pi]$

(16)   7. Suppose that $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - (x-1)^2 & \text{if } 0 \leq x \leq 2 \\ 0 & \text{if } x > 2 \end{cases}$.

a) Sketch $y = f(x)$ on the axes to the right:

b) Write a solution to the wave equation $u_{xx} = u_{tt}$ (here $c = 1$) for x in all of $\mathbb{R}$, the real numbers, with initial data $u(x,0) = f(x)$ and $u_t(x,0) = 0$. The formula in your answer will use the function $f(x)$.

c) Use your answer to b) to sketch $u(x, 3)$ on the axes given below.
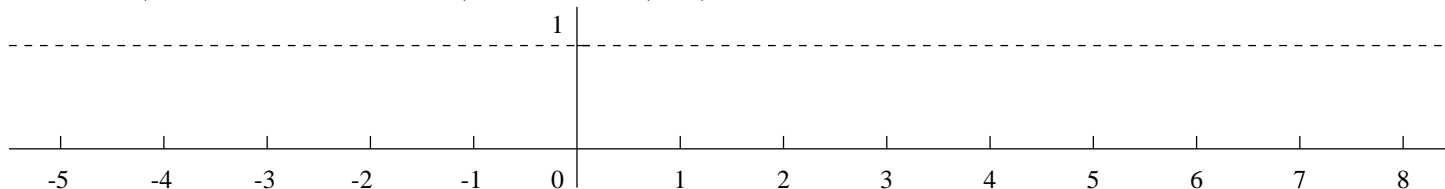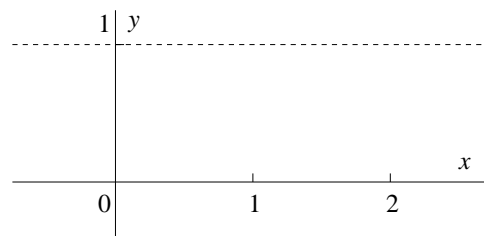
d) At approximately what positive time will the displacement at $x = 30$ first be greater than 0?

e) How many local maximums does the wave have at the time identified in d) and where are they located?

f) What is the velocity of the right-moving wave (in terms of space units per time units)?

(14)   8. Suppose $u(x, y, t)$ is a solution of the two-dimensional heat equation $u_t = u_{xx} + u_{yy}$ for $0 \le x \le \pi$ and $0 \le y \le \pi$ and $t \ge 0$. Also suppose that $u(x, y, t)$ satisfies the following boundary and initial conditions:
(BC) $u(0, y, t) = 0$ and $u(\pi, y, t) = 0$ for all y with $0 \le y \le \pi$ and for all non-negative t;
$u(x, 0, t) = 0$ and $u(x, \pi, t) = 0$ for all x with $0 \le x \le \pi$ and for all non-negative t;
(IC) $u(x, y, 0) = 6\sin(4x)\sin(7y) - 3\sin(5x)\sin(8y)$ for all x and y with $0 \le x, y \le \pi$.
a) Write a formula for $u(x, y, t)$.
b) Compute $E = \displaystyle\int_0^\pi \int_0^\pi \big(u(x, y, t)\big)^2 \, dx \, dy$. Use *orthogonality* to simplify the result.
c) E is a function of time, t. What happens to E as $t \to +\infty$?

(20)   9. Consider the following boundary value problem for $u(x, y)$, a function of two variables:
(PDE) $u_{xx} + u_{yy} + u = 0$.   (This is one form of the Helmholtz equation.)
(BC) $u(0, y) = 0$ and $u(\pi, y) = 0$ for all y.
a) Use separation of variables to find *non-zero* product solutions $X(x)Y(y)$ to this boundary value problem. Be explicit about the resulting eigenvalues and eigenfunctions.
b) Now suppose the specifications of the boundary value problem include the following initial conditions:
(IC) $u(x, 0) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} < x \le \pi \end{cases}$ <u>and</u> $u_y(x, 0) = 0$ for all x in $[0, \pi]$.
Use the product solutions found in a) together with the principle of superposition (linearity) to write a solution to the (PDE)+(BC)+(IC). Be as specific as you can about the resulting infinite series. In particular, write the first three terms (*not* just the coefficients!) of the series as explicitly as possible.

(18)   10. Suppose $u(x, t)$ is a solution of the wave equation $u_{xx} = u_{tt}$ (here $c = 1$) for $0 \le x \le \pi$. Additionally, suppose that $u(x, t)$ satisfies the boundary conditions
(BC) $u_x(0, t) = 0$ for all t and $u_x(\pi, t) = 0$ for all t.
and also the initial conditions
(IC) $u(x, 0) = f(x)$ where $f(x)$ is a function defined for $0 \le x \le \pi$ with $f'(0) = 0$ and $f'(\pi) = 0$, <u>and</u> $u_t(x, 0) = 0$ for all x between 0 and π.
Find a formula for $u(x, t)$ as a series whose terms are certain Fourier coefficients of $f(x)$ multiplied by trigonometric functions.
Comment The solution is similar to but *not* the same as those on the formula sheet. Separate variables, find eigenvalues and eigenfunctions, and use superposition (linearity).

(12)   11.   A bar of length π whose lateral surface is insulated is placed on the interval $[0, \pi]$. One end of the bar (at $x = 0$) is maintained at temperature 2 (so $u(0, t) = 2$ for all t) and the other end (at $x = \pi$) is insulated (so $u_x(\pi, t) = 0$ for all t). If $u(x, t)$ is the temperature in the bar at position x at time t, we suppose that the temperature satisfies the heat equation $u_{xx} = u_t$ (here $k = 1$) for all x in $[0, \pi]$ and $t \geq 0$. An initial temperature distribution $u(x, 0)$ is given as shown in the graph to the right. The graph shows a piecewise linear function connecting $(0, 2)$ and $(1, 1)$ and $(2, 3)$ and $(\pi, 3)$.
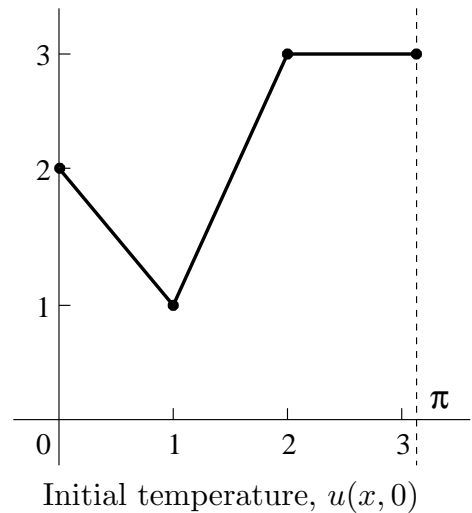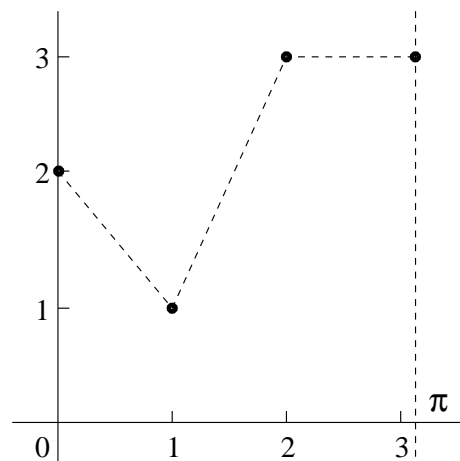
Suppose $u(x, t)$ is the solution to this boundary value problem.

Initial temperature, $u(x, 0)$

Comment We did not discuss this situation in class. Think of an appropriate steady state solution, and about how the initial temperature distribution would evolve toward it.

a) Sketch a graph of $u(x, \frac{1}{100})$ on the axes given below.
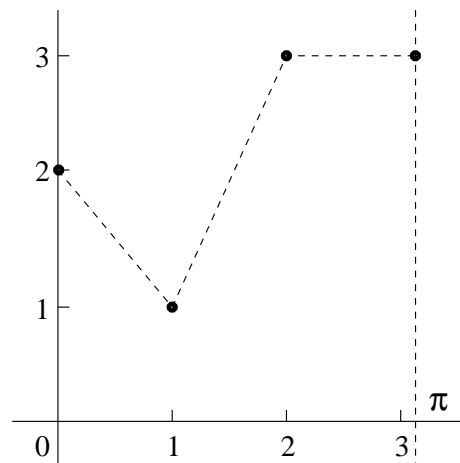
b) Sketch a graph of $u(x, 100)$ on the axes given below.

Final Exam for Math 421, section 1

December 16, 2005

NAME _____

Do all problems, in any order.

Show your work. An answer alone may not receive full credit.

No notes other than the distributed formula sheet may be used on this exam.

No calculators may be used on this exam.

Problem Number	Possible Points	Points Earned:
1	18	
2	18	
3	12	
4	18	
5	12	
6	16	
7	16	
8	14	
9	20	
10	18	
11	12	
Total Points Earned:		

Yes, the total number of points is 174. Please do the exam.