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Fourier series & PDE formulas

For $f(x)$ defined in $[-L, L]$, the Fourier series of $f(x)$ is $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right)+b_n \sin\left(\frac{\pi n x}{L}\right)$.

Fourier coefficients
$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$; $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi n x}{L}\right)dx$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{\pi n x}{L}\right)dx$ for $n > 0$.

Parseval's formula
$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) = \frac{1}{L}\int_{-L}^{L} f(x)^2\,dx$.

Orthogonality
If m and n are positive integers, then $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$ and $\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$ and $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0$ for all n and m.
If n is a positive integer, then $\int_{-\pi}^{\pi} \cos(0x)\cos(nx)\,dx = \begin{cases} 0 & \text{if } n \neq 0 \\ 2\pi & \text{if } n = 0 \end{cases}$ and $\int_{-\pi}^{\pi} \cos(0x)\sin(nx)\,dx = 0$ for all n and $\int_{-\pi}^{\pi} \sin(0x) \begin{Bmatrix} \cos(nx) \\ \sin(nx) \end{Bmatrix} dx = 0$ for all n.

The wave equation
Consider this initial/boundary value problem for the wave equation (c is *signal speed*):
$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ for x in $[0, L]$; $y(0,t) = y(L,t) = 0$; $y(x,0) = f(x)$; $\frac{\partial y(x,0)}{\partial t} = g(x)$.

Initial displacement only
$g(x) = 0$; $f(x)$ given: $y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{nc\pi t}{L}\right)$ with $c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi n x}{L}\right)dx$.

Initial velocity only
$f(x) = 0$; $g(x)$ given: $y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{nc\pi t}{L}\right)$ with $c_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{\pi n x}{L}\right)dx$.

D'Alembert solution
On all of $\mathbb{R}$, $y(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz$.

The heat equation
k is *diffusivity* and $[0, L]$ represents a bar with insulated sides: $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$.

Zero boundary conditions
$u(0,t) = u(L,t) = 0$; $u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$ with $c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ for $n > 0$.

Insulated ends
$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(L,t)}{\partial x} = 0$; $u(x,t) = \frac{c_0}{2}+\sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$ with $c_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ for $n \geq 0$.