Fourier series & PDE formulas

For $f(x)$ defined in $[-L, L]$, the Fourier series of $f(x)$ is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{\pi n x}{L}\right) + b_n \sin\left(\frac{\pi n x}{L}\right)$.

Fourier coefficients

$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$; $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{\pi n x}{L}\right) dx$ and $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{\pi n x}{L}\right) dx$ for $n > 0$.

Parseval's formula

$\frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \frac{1}{L} \int_{-L}^{L} f(x)^2\, dx$.

Orthogonality

If m and n are positive integers, then $\int_{-\pi}^{\pi} \sin(mx) \sin(nx)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$ and

$\int_{-\pi}^{\pi} \cos(mx) \cos(nx)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$ and $\int_{-\pi}^{\pi} \sin(mx) \cos(nx)\, dx = 0$ for all n and m.

If n is a positive integer, then $\int_{-\pi}^{\pi} \cos(0x) \cos(nx)\, dx = \begin{cases} 0 & \text{if } n \neq 0 \\ 2\pi & \text{if } n = 0 \end{cases}$ and

$\int_{-\pi}^{\pi} \cos(0x) \sin(nx)\, dx = 0$ for all n and $\int_{-\pi}^{\pi} \sin(0x) \begin{Bmatrix} \cos(nx) \\ \sin(nx) \end{Bmatrix} dx = 0$ for all n.

The wave equation

Consider this initial/boundary value problem for the wave equation (c is *signal speed*):
$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ for x in $[0, L]$; $y(0, t) = y(L, t) = 0$; $y(x, 0) = f(x)$; $\frac{\partial y(x, 0)}{\partial t} = g(x)$.

Initial displacement only

$g(x) = 0$; $f(x)$ given: $y(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n \pi x}{L}\right) \cos\left(\frac{n c \pi t}{L}\right)$ with $c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n x}{L}\right) dx$.

Initial velocity only

$f(x) = 0$; $g(x)$ given: $y(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n \pi x}{L}\right) \sin\left(\frac{n c \pi t}{L}\right)$ with $c_n = \frac{2}{n \pi c} \int_0^L g(x) \sin\left(\frac{\pi n x}{L}\right) dx$.

D'Alembert solution

On all of $\mathbb{R}$, $y(x, t) = \frac{1}{2} \left(f(x + ct) + f(x - ct)\right) + \frac{1}{2c} \int_{x - ct}^{x + ct} g(z)\, dz$.

The heat equation

k is *diffusivity* and $[0, L]$ represents a bar with insulated sides: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$.

Zero boundary conditions

$u(0, t) = u(L, t) = 0$; $u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n \pi x}{L}\right) e^{-k\left(\frac{n \pi}{L}\right)^2 t}$ with $c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n \pi x}{L}\right) dx$ for $n > 0$.

Insulated ends

$\frac{\partial u(0, t)}{\partial x} = \frac{\partial u(L, t)}{\partial x} = 0$; $u(x, t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n \pi x}{L}\right) e^{-k\left(\frac{n \pi}{L}\right)^2 t}$ with $c_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n \pi x}{L}\right) dx$ for $n \geq 0$.

The other side has some Laplace transform formulas, and some trig. & hyperbolic formulas.

Laplace transforms

Function	Laplace Transform
$f(t)$	$F(s) = \displaystyle\int_0^\infty e^{-st} f(t)\, dt$
$af(t) + bg(t)$	$aF(s) + bG(s)$
t^n (positive integer n)	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$
$\sin(kt)$	$\dfrac{k}{s^2 + k^2}$
$\cos(kt)$	$\dfrac{s}{s^2 + k^2}$
$e^{at} f(t)$	$F(s-a)$
$\mathcal{U}(t-a) f(t-a)$	$e^{-as} F(s)$
$g(t)\mathcal{U}(t-a)$	$e^{-as}\mathcal{L}\{g(t+a)\}$
$f'(t)$	$sF(s) - f(0^+)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$
$(f * g)(t) = \displaystyle\int_0^t f(t-\tau) g(\tau)\, d\tau$	$F(s)G(s)$
$\delta(t-a)$	e^{-as}
$\displaystyle\int_0^t f(w)\, dw$	$\dfrac{1}{s} F(s)$
$t^n f(t)$	$(-1)^n \dfrac{d^n}{ds^n} F(s)$
$f(t+T) = f(t)$ (periodic)	$\dfrac{1}{1 - e^{-sT}} \displaystyle\int_0^T e^{-st} f(t)\, dt$

Some useful facts

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$e^{iA} = \cos A + i \sin A$$
$$\cosh A = \frac{e^A + e^{-A}}{2} \text{ and } \sinh A = \frac{e^A - e^{-A}}{2}$$

The other side presents some Fourier series and PDE formulas.