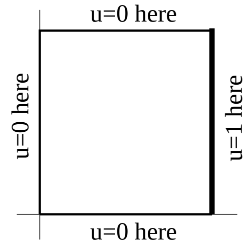
Math 421

An example of two-dimensional heat flow

December 5

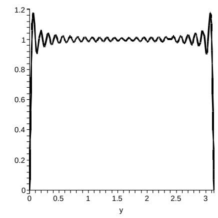


I want *steady-state solutions* for the two-dimensional heat equation, $u_{tt} = u_{xx} + u_{yy}$, in one the square which has sides parallel to the coordinate axes and each side π units long, with lower-left hand corner is at the origin, $(0, 0)$. Since u is supposed to be a steady-state solution, $u_t = 0$ always, and we can omit the t in the variables we give u. We are actually looking for solutions $u(x, y)$ to *Laplace's equation*, $u_{xx} + u_{yy} = 0$ in the π-by-π square. The boundary conditions are:

(BC) $u(x, 0) = 0$ & $u(x, \pi) = 0$ for $0 \le x \le \pi$; $u(0, y) = 0$ & $u(\pi, y) = 1$ for $0 \le y \le \pi$

Here are `Maple` commands to generate a partial sum of the Fourier sine series for the function 1 (a function which is always 1):

```
>c:=n->(2/Pi)*int(1*sin(n*y),y=0..Pi);
>plot(sum(c(j)*sin(j*y),j=1..50),y=0..Pi,thickness=2,color=black);
```



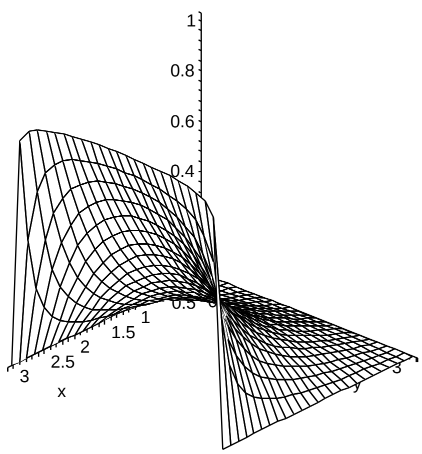
As can be expected, the graph is all fuzzy at the ends (Gibb's phenomenon again). Now we can try to look at a partial sum of the solution to Laplace's equation:

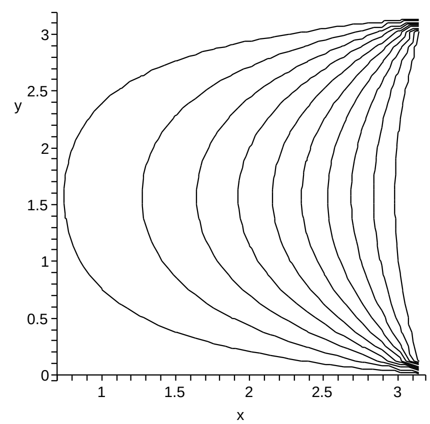
```
>u:=(x,y)->sum((1/sinh(j*Pi))*c(j)*sin(j*y)*sinh(j*x),j=1..50);
```

`Maple` reports that $u(1., 2.)$ is .1176183537. We can draw some pictures with these commands:

```
>plot3d(u(x,y),x=0..Pi,y=0..Pi,axes=normal);
>contourplot(u(x,y),x=0..Pi,y=0..Pi,contours=10,color=black);
```

Below to the left is a picture of the surface $z = u(x, y)$. On the right is a contour plot of $u(x, y)$:





Here are some slices of this surface by planes perpendicular to the xy-plane. The commands were:

```
>plot({u(x,.1),u(x,.3),u(x,.5)},x=0..Pi,color=black,thickness=2);
>plot({u(.1,y),u(.3,y),u(.7,y)},y=0..Pi,color=black,thickness=2);
```

Which slices are which curves?