Math 421:01 # One heat flow example **November 21, 2005**

The `Maple` command which follows defines a function piecewise. In the language of Laplace transforms, $F(x) = \mathcal{U}(x - \frac{\pi}{3}) - \mathcal{U}(x - \frac{\pi}{2})$. It is a block of height 1 in the interval $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ and is 0 otherwise.

```
>F:=x->piecewise(x<Pi/3,0,x<Pi/2,1,0);
```

I'll use this as initial data for the heat equation with temperature at the ends always 0. So we're looking for $u(x,t)$ with $u_t = u_{xx}$ and, for all t, both $u(0,t) = 0$ and $u(\pi,t) = 0$.

Interpretation Would you believe a thin bar has temperature 0 except for a central chunk which has temperature equal to 1? This seems physically unlikely. This is more possibly an initial condition for diffusion, say a sugar solution in water. Adjust the units for concentration so that the highest concentration expected is 1 and a solution entirely water has concentration 0. Then we've considering a thin tube of water which has a high concentration of sugar in a central interval: maybe this is possible.

Separation of variables suggests that we compute the Fourier series of the function, F. We want a Fourier sine series which will be valid on the interval $[0, \pi]$. This `Maple` command gets the Fourier sine coefficients:

```
>g:=n->(2/Pi)*int(F(x)*sin(n*x),x=0..Pi);
```

Let's check:

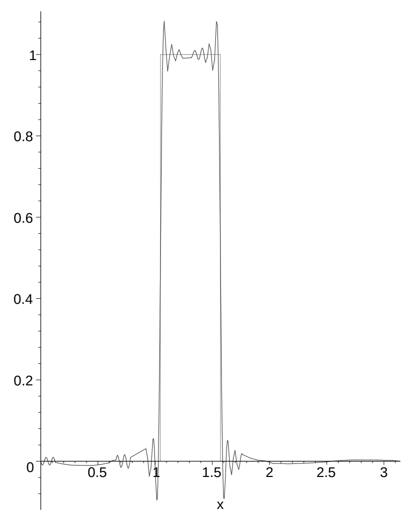
```
>g(3);
                       2
                   - ----
                     3 Pi
```

The following instruction assembles a partial sum of the Fourier sine series for F:

```
>Q:=(N,t)->sum(g(n)*sin(n*x),n=1..N);
```

We can check it:

```
>Q(3);
            sin(x)       sin(2 x)       sin(3 x)
            ------ + 1/2 -------- - 2/3 --------
              Pi            Pi             Pi
```



To the right is a picture of the 100^{th} partial sum for F and F itself. The graph results from the `Maple` command

```
>plot({F(x),Q(100)},x=0..Pi);
```

Notice that `Maple` attempts to fit the graph into a square. The true aspect ratio (horizontal : vertical) is actually about 3-to-1.

Now let's look at graphs of solutions to the heat equation, approximating the initial data given by F on $[0, \pi]$ with the zero boundary conditions. So we need a slight variation of the partial Fourier sum defined above. Here it is:

```
>QQ:=(N,t)->sum(g(n)*sin(n*x)*exp(-n^2*t),n=1..N);
```

And here is a test:

```
>QQ(3,t);
        sin(x) exp(-t)       sin(2 x) exp(-4 t)       sin(3 x) exp(-9 t)
        -------------- + 1/2 ------------------ - 2/3 ------------------
              Pi                     Pi                       Pi
```

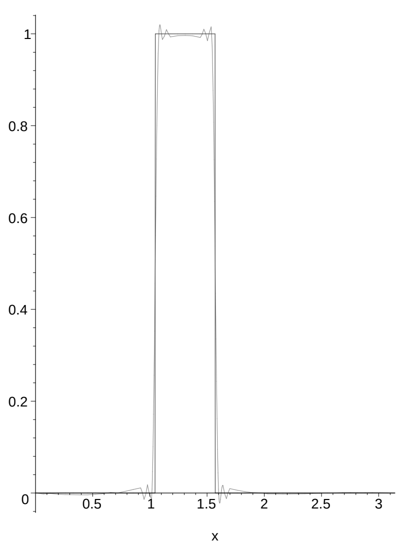
If this is too complicated, we can look at a fixed value of t:

```
>QQ(3,.01);
       0.3151426498 sin(x) + 0.1529143885 sin(2.x) - 0.1939422210 sin(3.x)
```

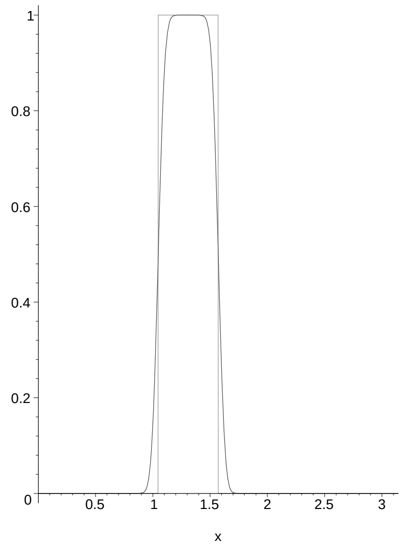
On the next page is the result of the following command for various values of t:

```
>plot({F(x),QQ(100,t)},x=0..Pi);
```

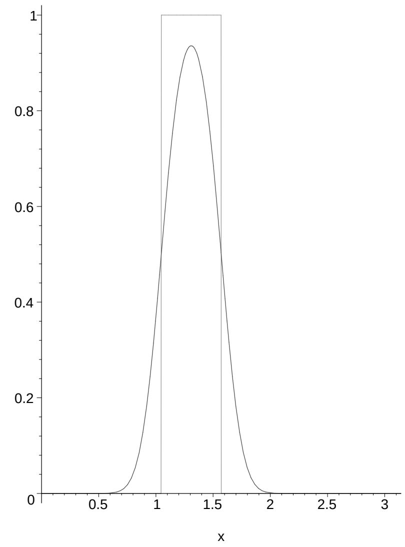
OVER



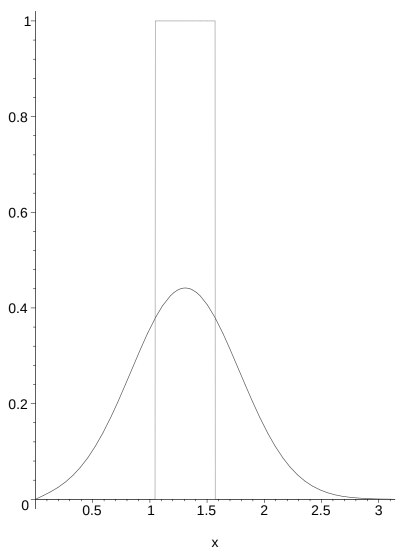
t=.0001



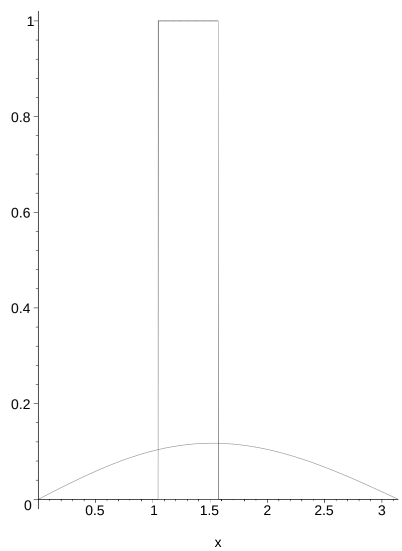
t=.001



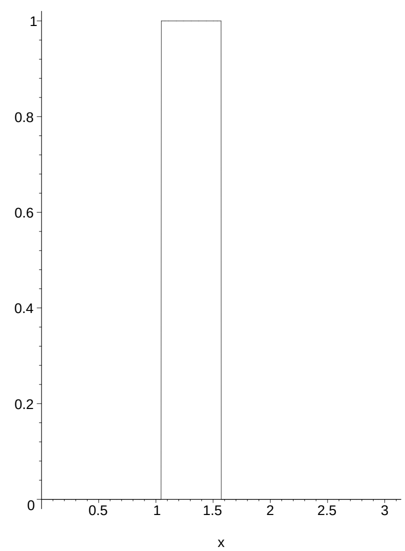
t=.01



t=.1



t=1



t=10