Final exam # Math 503 **December 17, 2004**

Please do all problems.

1. Suppose that $H = \{z \in \mathbb{C} : \text{Im } z > 0\}$, the open upper half-plane, and suppose that $\mathcal{F} = \{g : H \to H,\ g \text{ is holomorphic, } \& \ g(i) = i\}$, the set of holomorphic mappings from H to H which fix i. Prove that $M = \sup\{|g'(i)| \text{ for } g \in \mathcal{F}\}$ is finite. Find all g's in $\mathcal{F}$ with $|g'(i)| = M$.

2. Use the Residue Theorem to compute $\displaystyle\int_0^{2\pi} \sqrt{2 + \cos\theta + \sin\theta}\, d\theta$.

3. Prove that the sum

$$S(z) = \sum_{n=1}^{\infty} \frac{\sin\left(\frac{z}{n}\right)}{n}$$

converges for all $z \in \mathbb{C}$ and that $S(z)$ is holomorphic in $\mathbb{C}$. What is $S'(0)$?

Note You may use the result verified by Mr. Trainor in his presentation: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. State clearly any theorems you need, and carefully verify any hypotheses of these theorems.

4. If f is an entire function which maps every unbounded sequence to an unbounded sequence, then f is a polynomial.

5. Let $\Omega = \{z \in \mathbb{C} : 0 < |z| < \infty\}$. Determine all holomorphic functions f on Ω such that

$$|f(z)| < \frac{1}{|z|^{1/2}} + |z|^{1/2}, \qquad z \in \Omega.$$

Justify your answer.