**Homework #1** **Math 503** **September 1, 2004**

**Due Wednesday, September 8, 2004**

Please read §1.1 (pp. 3–10) in $\boldsymbol{N^2}$ (my label for our text, *Complex Analysis in One Variable*, Second Edition, by Narasimhan and Nievergelt) so that you can help me with the lectures. Also please read all of the **Exercises 1–36** for Chapter 0 (pp. 259-266). Please hand in the following problems on Monday, September 6.

Problem 1: **Exercise 7** and **Exercise 8** state important inequalities which we will use many times. For all $z, w \in \mathbb{C}$, $|w+z| \leq |w| + |z|$ and $|w-z| \geq \big||w| - |z|\big|$. The first inequality is called the Triangle Inequality and the second is frequently called the Reverse Triangle Inequality. I hope you have seen them before and can prove them.

But a *reverse* inequality *could* be $|w+z| \overset{?}{\geq} |w| + |z|$. This inequality can be false ($z = -1$ and $w = 1$) but is sometimes true (when $z$ and $w$ are both positive real numbers, for example). Maybe we can *help* it to be true by putting in a magnification factor.

> Prove that there is a connected open neighborhood $S$ in $\mathbb{C}$ of the positive real numbers so that for all $z, w \in S$, $3|w+z| \geq |w| + |z|$.

Problem 2: If $v$ and $w$ are non-zero vectors in $\mathbb{R}^2$, then the angle between $v$ and $w$ is $\arccos\left(\frac{v \cdot w}{\|v\|\,\|w\|}\right)$. A linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ *preserves angles* if $T$ is 1-to-1 and if, for all non-zero vectors $v$ and $w$ in $\mathbb{R}^2$, the angle between $v$ and $w$ is the same as the angle between $T(v)$ and $T(w)$. Describe matrix representations for all linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ which preserve angles. Further describe matrix representations for all linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ which preserve angles and are orientation-preserving ($\det(T) > 0$).

Problem 3: Suppose that $u(x,y)$ and $v(x,y)$ are $C^1$ real-valued functions defined on all of $\mathbb{R}^2$. Let $F(z)$, where $z = x + iy$, be defined by $F(z) = u(x,y) + iv(x,y)$.

a) Can you find $u$ and $v$ so that $F$ is $\mathbb{C}$-differentiable only when $(x,y) = (0,0)$ (the origin)? If yes, display and verify an example. If no, explain why not.

b) Can you find $u$ and $v$ so that $F$ is $\mathbb{C}$-differentiable only when $x \geq 0$ (the closed right half-plane)? If yes, display and verify an example. If no, explain why not.

c) Can you find $u$ and $v$ so that $F$ is $\mathbb{C}$-differentiable only when $(x,y) \neq (0,0)$ (the complement of the origin)? If yes, display and verify an example. If no, explain why not.

Problem 4: Do **Exercise 24**, which follows.

> For each $c \in D(0,1)$ define a fractional linear transformation $L_c$ by $L_c(z) := \dfrac{z - c}{1 - \overline{c}z}$. Prove such a fractional linear transformation maps the unit disc onto the unit disc, and the unit circle onto the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$.