Homework #2 **Math 503** **September 13, 2004**

Due Wednesday, September 22, 2004

Please read §1.2 (pp. 10–22) in $\boldsymbol{N^2}$.

Problem 1: Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of non-negative real numbers and $\{c_n\}$ is the sequence defined by $c_n = a_n b_n$. Let $A = \limsup a_n$, $B = \limsup b_n$ and $C = \limsup c_n$. If $\{a_n\}$ converges, prove that $C = AB$. Show by example that if we do not assume $\{a_n\}$ converges, then C and AB may not be equal. What relationship must hold between C and AB whether or not convergence is assumed, and why?

Problem 2: Show that the series

$$\frac{1}{1+|z|} - \frac{1}{2+|z|} + \frac{1}{3+|z|} - \frac{1}{4+|z|} + \cdots + \frac{(-1)^{n-1}}{n+|z|} + \cdots$$

is not absolutely convergent but is uniformly convergent in the whole complex plane.

From *Classical Complex Analysis* by Liang-sin Hahn and Bernard Epstein.

Problem 3: Suppose that $\varphi(r)$ is a function defined for $r \geq 0$ which is bounded in every finite interval and tends to ∞ as $r \to \infty$. Prove that there is an entire function*, F, which is real on $\mathbb{R}$ so that $F(r) \geq \varphi(r)$ for all real $r \geq 0$.

Hint You may consider an everywhere convergent power series $\sum_{n=1}^{\infty} \left(\frac{z}{n}\right)^{\lambda_n}$ for a "sufficiently rapidly increasing sequence of positive integers", $\{\lambda_n\}$.

From *Analytic Functions* by Stanisław Saks and Antoni Zygmund, and attributed to Poincaré.

Problem 4: Prove that if $|z| < 1$ then

$$\sum_{n=1}^{\infty} \frac{z^n}{1-z^n} = \sum_{m=1}^{\infty} d(m) z^m$$

where $d(m)$ is the number of divisors of the positive integer m. Also, prove that both series converge uniformly on compact subsets of $D(0,1)$, the unit disc.

The *generating function* of $\{d(m)\}$, Sequence A000005 in Sloane's `On-Line Encyclopedia of Integer Sequences`.

Problem 5: Show that the power series

$$\frac{z^3}{1} - \frac{z^{2\cdot 3}}{1} + \frac{z^{3^2}}{2} - \frac{z^{2\cdot 3^2}}{2} + \cdots + \frac{z^{3^n}}{n} - \frac{z^{2\cdot 3^n}}{n} + \cdots$$

has radius of convergence 1, and that the points of convergence and those of divergence of this series each form sets which are everywhere dense in $\partial D(0,1)$.

Hint Take points of the form $z = \exp\left(\frac{\pi i k}{3^N}\right)$ and consider the case of k odd and k even.

From *Analytic Functions* by Stanisław Saks and Antoni Zygmund, and attributed to Vijayaraghavan.

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* A function which is holomorphic in all of $\mathbb{C}$ is called *entire*.

Problem 6: Do **Exercise 48** of $\boldsymbol{N^2}$, which follows.

For each function $f\colon \Omega \to \mathbb{C}$ holomorphic on a *connected* open set $\Omega \subseteq \mathbb{C}$ prove the following statements.

(48.1) If $f'(z) = 0$ for every $z \in \Omega$, then f is constant.

(48.2) If there exists $c \in \mathbb{C}$ such that $f(z) = c \cdot \overline{f(z)}$ for every $z \in \Omega$, then f is constant.

(48.3) If $f(\Omega) \subseteq \mathbb{R}$, then f is constant.

(48.4) If $|f|$ is constant, then f is constant.

(48.5) If $g\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and $g \circ f$ is constant, then f or g is constant.

(48.6) If $f_1, f_2, \ldots, f_N$ are holomorphic in Ω and if $|f_1|^2 + |f_2|^2 + \cdots ||f_n|^2$ is constant, then each f_j is constant.