Homework #3 **Math 503** **September 27, 2004**

Due Wednesday, October 6, 2004

Please read §1.3 and 1.4 (pp. 22–36) in $\boldsymbol{N^2}$.

Problem 1: Do **Exercise 73** of $\boldsymbol{N^2}$, which follows:

One of the definitions of the **dilogarithm** Li_2 is the series $Li_2(z);=\sum_{n=1}^{\infty}\frac{z^n}{n^2}$.

(73.1) Determine its radius of convergence, R.

(73.2) Determine whether the series converges on the closure $\overline{D(0,R)}$.

(73.3) For each real $\varepsilon>0$, determine an integer n_ε such that for every integer $m\geq n_\varepsilon$ and for every $z\in D(0,R)$, $\left|\sum_{n>m}\frac{z^n}{n^2}\right|<\varepsilon$.

(73.4) Show that inside the topological interior of the disc of convergence the complex dilogarithm satisfies a second-order linear ordinary differential equation with rational coefficients.

Problem 2: Suppose μ is a compactly supported measure in $\mathbb{C}$. Define F_μ, the *Cauchy transform* of μ, by $F_\mu(z)=\int_{\mathbb{C}}\frac{1}{w-z}\,d\mu_w$ for $z\notin\operatorname{supp}\mu$.

a) Prove that F_μ is holomorphic in $\mathbb{C}\setminus\operatorname{supp}\mu$ and that $\lim_{z\to\infty}F_\mu(z)=0$*.

b) Suppose that μ is Lebesgue measure on the boundary of the unit circle. What is F_μ?

c) Suppose that μ is Lebesgue measure on the unit interval, $[0,1]$, of $\mathbb{R}$. What is F_μ?

Problem 3: Show that the series $\sum_{n=1}^{\infty}\frac{z}{(1+|z|)^n}$ converges (absolutely) pointwise but not locally uniformly on $\mathbb{C}$.

From *Classical Complex Analysis* by Liang-sin Hahn and Bernard Epstein.

The following two problems are from *Theory of Complex Functions* by Reinhold Remmert.

Problem 4: Using the Cauchy integral formula calculate

a) $\int_{\partial D(0,2)}\frac{e^z\,dz}{(z+1)(z-3)^2}$

b) $\int_{\partial D(0,2)}\frac{\sin z}{z+i}\,dz$
$\left(\sin(z)\text{ is }\frac{e^{iz}-e^{-iz}}{2i}\text{ or anything convenient.}\right)$

c) $\int_{\partial D(-2i,2)}\frac{dz}{z^2+1}$

d) $\int_{\partial D(0,1)}\frac{e^z\,dz}{(z-2)^3}\,dz$

Problem 5: Let f be holomorphic in $D(0,R)$, $R>1$. Calculate the integrals $\int_{\partial D(0,1)}\left(2\pm(\zeta+\zeta^{-1})\right)\frac{f(\zeta)}{\zeta}\,d\zeta$ two different ways and thereby deduce that
$\pi^{-1}\int_0^{2\pi}f(e^{it})\cos^2\left(\frac{1}{2}t\right)dt=f(0)+\frac{1}{2}f'(0)$ and $\pi^{-1}\int_0^{2\pi}f(e^{it})\sin^2\left(\frac{1}{2}t\right)dt=f(0)-\frac{1}{2}f'(0)$.

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* Given $\varepsilon>0$, there is $M>0$ so that if $|z|>M$ then $F_\mu(z)$ is defined and $|F_\mu(z)|<\varepsilon$.