Homework #4 **Math 503**[♡] **October 6, 2004**

Due Wednesday, October 20, 2004

Please finish reading §1.4 and §1.5 (pp. 32–43) in $\boldsymbol{N^2}$. Then we should begin chapter 2.

Problem 1: Suppose that f is holomorphic in $D(0,1)$ and that for all sufficiently large integers, n, $\left|f\left(\frac{1}{n}\right)\right| \le \frac{1}{n!}$. Prove that f is the zero function.

Problem 2: a) An entire function is of *exponential type* if there are $A > 0$ and $C > 0$ so that for all $z \in \mathbb{C}$ with $|z| > C$, $|f(z)| \le Ce^{A|z|}$. Prove that the collection of entire functions of exponential type is closed under differentiation. In fact, if f is of exponential type, so is f', with the same A but with a possibly different C.

b) Show that an analogous statement for C^∞ functions on $\mathbb{R}$ is false. That is, define a C^∞ real-valued function g on $\mathbb{R}$ to be of *exponential type* if there are $A > 0$ and $C > 0$ so that for all $x \in \mathbb{R}$ with $|x| > C$, $|f(x)| \le Ce^{A|x|}$. Give an example of a C^∞ function on $\mathbb{R}$ of exponential type whose derivative is **not** of exponential type for *any* choices of C and A.[♢]

Problem 3: Suppose U is open in $\mathbb{R}$. A function $f\colon U \to \mathbb{R}$ is *real analytic* or C^ω if, for every $a \in U$, there is $\delta > 0$ with $(a-\delta, a+\delta) \subseteq U$ and there is a sequence of real numbers $\{c_n\}$ so that $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all $x \in (a-\delta, a+\delta)$: f is <u>real</u> analytic if f is locally the sum of a <u>real</u> power series.

a) Suppose $F\colon \Omega \to \mathbb{C}$ is holomorphic and $F(\Omega \cap \mathbb{R}) \subseteq \mathbb{R}$. If $U = \Omega \cap \mathbb{R}$ and $f = F\big|_U$, prove that $f\colon U \to \mathbb{R}$ is real analytic.

b) Suppose $f\colon U \to \mathbb{R}$ is real analytic with U open and connected in $\mathbb{R}$. Prove that there is an open and connected subset Ω of $\mathbb{C}$ and a complex analytic function $F\colon \Omega \to \mathbb{C}$ so that $U = \Omega \cap \mathbb{R}$ and $f = F\big|_U$.[♣]

c) Find an f which is C^∞ on $\mathbb{R}$ but *not* real analytic on $\mathbb{R}$.[♢]

d) Find an f which is real analytic on $\mathbb{R}$ whose Taylor series at 0 has finite radius of convergence.[♢]

e) Liouville's Theorem is false for real analytic functions: find an f which is real analytic on $\mathbb{R}$, non-constant, and bounded.[♢] Why does such an example exist?

f) An "entire" real analytic function may not have a complexification with domain $\mathbb{C}$. If $\varepsilon > 0$ define $S_\varepsilon \subset \mathbb{C}$ by $S_\varepsilon = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \varepsilon\}$. Construct a real analytic function $f\colon \mathbb{R} \to \mathbb{R}$ so that there is *no* holomorphic F_ε defined on S_ε with $F_\varepsilon\big|_{\mathbb{R}} = f$ for *any* $\varepsilon > 0$.[♢]

Problem 4: a) Read a version of the Weierstrass Approximation Theorem. The theorem statement should begin: "If f is continuous and real-valued on $[a,b] \subset \mathbb{R}$ and if $\varepsilon > 0$, then there is $P(x) \in \mathbb{R}[x]$ so that ..."

b) If $P(z) \in \mathbb{C}[z]$, prove that $\sup\{\left|\frac{1}{z} - P(z)\right| : z \in \partial D(0,1)\} \ge 1$.

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♡ Thanks to Mr. Nguyen for suggesting these wonderful footnote characters.

♣ F is called a *complexification* of f.

♢ Easy examples are better than complicated examples. I think there are easy examples for 2b), 3c), 3d), 3e), and 5b). Maybe there's an easy example for 3f) also. I would like that.

Problem 5: A holomorphic function f has a *holomorphic log* if there is g, holomorphic in f's domain, so that $e^g = f$. A holomorphic function f has a *holomorphic square root* if there is g, holomorphic in f's domain, so that $g^2 = f$. Let $D^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$.

a) If f is holomorphic and has a holomorphic log, then f has a holomorphic square root.

b) Give an example of a non-zero holomorphic function defined on D^* which has a square root but does not have a log.$\diamondsuit$ Previous page!

c) Prove that that z has no holomorphic square root in D^*.♠

Problem 6: a) Use what we've done so far in the course to find a biholomorphic mapping of the first quadrant to the unit disc. What is the image of the half line $\{y = x\} \cap \{x > 0\}$? What is the image of the quarter circle $\{|z| = 1\} \cap \{0 < \arg z < \frac{\pi}{2}\}$?

b) Use what we've done so far in the course to find a biholomorphic mapping of the strip $\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ to the unit disc. What is the image of the line segment $\{0 < \operatorname{Re} z < 1\} \cap \{\operatorname{Im} z = 0\}$? What is the image of the line $\{\operatorname{Re} z = \frac{1}{2}\}$?

♠ We did verify that z has no log in D^* (if it did, then the log's derivative is $\frac{1}{z}$ which has no primitive in D^*). But because of b), a simple converse to a) is not likely!