Homework #5 **Math 503** **October 25, 2004**

Due Wednesday, November 3, 2004

Problem 1: Suppose f is holomorphic in the unit disc, and $f\left(\frac{\sqrt{-1}}{k}\right) = \frac{100}{k^4}$ for integer $k \geq 2$. What is f exactly?

From a qualifying exam at the University of California–Irvine

Problem 2: Let f be a function defined in a domain D in $\mathbb{C}$ such that $f(z)^3$ is holomorphic in D.

a) Is $f(z)$ holomorphic in D? (Prove or give a counterexample to your answer.)[♡]

b) If $f \in C^1(D)$, can you conclude that f is holomorphic in D? (Verify your answer.)

From a qualifying exam at the University of California–Irvine

Problem 3: Without using the Fundamental Theorem of Algebra, prove for any P of degree $n \geq 1$ that

$$\lim_{R\to\infty} \int_{\partial D(0,R)} \frac{P'(z)}{P(z)}\, dz = 2\pi i\, n.$$

Deduce the Fundamental Theorem of Algebra from this equality.

From a qualifying exam at the University of Illinois

Problem 4: Suppose that U is an open subset of $\mathbb{C}$, and $\{K_n\}_{n\in\mathbb{N}}$ is a sequence of compact sets so that $\displaystyle\bigcup_{n\in\mathbb{N}} K_n = U$ and $K_n \subseteq \text{interior}(K_{n+1})$ for all $n \in \mathbb{N}$. Let $C(U)$ be the vector space of continuous complex-valued functions defined on U. If f and g are in $C(U)$, define $d(f,g)$ by

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{K_n}}{1+\|f-g\|_{K_n}}$$

where $\|h\|_S = \sup\{|h(z)| : z \in S\}$. Prove that $(C(U), d)$ is a complete metric space, and that $f_j \xrightarrow{d} f$ if and only if $\{f_j\}$ converges uniformly to f on every compact subset of U.

Problem 5: Find all uniformly continuous entire functions.

From a qualifying exam at Purdue University

Problem 6: Suppose that f and g are entire functions satisfying $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that $f(z) = Cg(z)$ for some $C \in \mathbb{C}$.

This has appeared on written qualifying exams given by at least three different schools.

Problem 7: Suppose in this problem that f has an isolated singularity at 0.

a) Prove that if e^f has a removable singularity at 0, then f must have a removable singularity at 0.

b) Under what conditions does e^f have a pole at 0?

♡ This is how the question was phrased. I think the sentence is weird.