Homework #6 **Math 503** **November 17, 2004**

Due Monday, November 29, 2004

Problem 1: Suppose f is an entire function, and the Taylor series centered at 0 for f converges uniformly to f on $\mathbb{C}$. Prove that f is a polynomial.*

Problem 2. Let U be the region between the circles $|z-1|=1$ and $|z|=3$. Find a conformal mapping of an open neighborhood of $\overline{U}$ which carries U to an annular region $A(0,r,R)=\{z : r<|z|<R\}$.

Problem 3. Prove that for any fixed complex number ζ,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\zeta\cos\theta}\,d\theta=\sum_{n=0}^{\infty}\left(\frac{\zeta^n}{n!}\right)^2.$$

From a qualifying exam at the University of California–Berkeley

Problem 4: Suppose U is a simply connected open subset of $\mathbb{C}$, and f is a holomorphic function on U which is not identically 0. Find a condition on the zeros of f which is equivalent to the existence of holomorphic square root, g, of f on U. Prove this equivalence.**

Problem 5. Let $D=\{z\in\mathbb{C} : |z|<1\}$ be the open unit disc. Find the image of D under the map $f(z)=e^{\frac{i-iz}{z+1}}$.

From a qualifying exam at Johns Hopkins University

Problem 6: Suppose f is holomorphic in the unit disk, and f is not identically 0. Prove that there is a natural number n such that $f\left(\frac{1}{n}\right)\neq\frac{1}{n+2}$.

From a qualifying exam at the University of Iowa

Problem 7: Suppose S is the set $\{0\}\cup\{r=e^{\theta} : \theta\in\mathbb{R}\}$ (the second set is an *equiangular spiral* described in polar coordinates). S is closed, and $U=\mathbb{C}\setminus S$ is open. Try to convince me that U is simply connected.

Maybe this is part of what is "fun" in topology. Please do not spend too much time on this problem (it could take 10 pages, easily!). An attempted explanation would use the word compact somewhere, and would sort of describe a homotopy in some way. Sigh.

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* So uniform convergence on all of $\mathbb{C}$ is **wrong, wrong, wrong!**

** We will analyze this situation when f is never 0 on U in class. It is also discussed in section 2.5 of $\boldsymbol{N^2}$.