Homework #7 **Math 503** **November 29, 2004**

Due Monday, December 13, 2004

The following three problems are from Remmert's *Theory of Complex Functions*.

Problem 1: For $a > 1$ show that $\int_0^{2\pi} \frac{d\phi}{a+\sin\phi} = \frac{2\pi}{\sqrt{a^2-1}}$.

Problem 2: Prove the identity $\int_{-\infty}^{\infty} \frac{dx}{(x^4+a^4)^2} = \frac{3}{8}\frac{\sqrt{2}}{a^7}\pi$ for $a > 0$.

Problem 3: Prove that $\int_0^{\infty} \frac{\sqrt{x}}{x^2+a^2} dx = \frac{\pi}{\sqrt{2a}}$.

The following problem is Exercise 239 in $\boldsymbol{N^2}$. Similar problems are found on written qualifying exams of many universities.

Problem 4: For each positive integer n, and for each real $\lambda > 1$, prove that the equation

$$z^n = e^{z-\lambda}$$

has *no* solutions with $|z| = 1$, and exactly n simple solutions with $|z| < 1$.

A continuous function is *proper* if and only if the inverse image of every compact set is compact. The following problem is Exercise 297 in $\boldsymbol{N^2}$.

Problem 5: Prove that there is *no* proper holomorphic map from the open unit disc into the complex plane.

The following problem is from Conway's *Functions of One Complex Variable*.

Problem 6: Does there exist a holomorphic function $f : D(0,1) \to D(0,1)$ with $f\left(\frac{1}{2}\right) = \frac{3}{4}$ and $f'\left(\frac{1}{2}\right) = \frac{2}{3}$?

The following problem is from Remmert's *Theory of Complex Functions*. Here H is the open upper halfplane, so $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.

Problem 7: Let $f : D(0,1) \to H$ be holomorphic with $f(0) = i$. If $f(0) = i$, prove that

a) $\frac{1-|z|}{1+|z|} \le |f(z)| \le \frac{1+|z|}{1-|z|}$ for $z \in D(0,1)$.

b) $|f'(0)| \le 2$.

Problem 8: Let f be a holomorphic function which maps the unit disk into the unit disk. Show that $|f(z) + f(-z)| \le 2|z|^2$ for all z in the unit disk, and if the equality holds for some z, then $f(z) = e^{i\theta} z^2$ for some real θ.

From a qualifying exam at Johns Hopkins University†

† This semester I've worked with a study group of grad students who are preparing for our written exams. We looked at this problem. At the urging of the VERY KIND students in the study group, I advise you that the problem statement, copied directly from the Johns Hopkins exam, is incorrect. Please do one of the following two alternative problems.

1 Find a counterexample to the problem as stated. Then add a simple hypothesis to the problem which makes it correct, and solve the resulting problem.

2 Solve this problem, quoted from Remmert's *Theory of Complex Functions*. Here $\mathbb{E}$ is the unit disc.

> Let $f : \mathbb{E} \to \mathbb{E}$ be holomorphic, with $f(0) = 0$. Let $n \in \mathbb{N}$, $n \geq 1$, $\zeta := e^{2\pi i/n}$. Show that
>
> $$(*) \qquad |f(\zeta z) + f(\zeta^2 z) + \cdots + f(\zeta^n z)| \leq n|z|^n \quad \text{for all } z \in \mathbb{E}.$$
>
> Moreover, if there is at least one $c \in \mathbb{E} \setminus \{0\}$ such that equality prevails in (*) at $z = c$, then there exists an $a \in \partial\mathbb{E}$ such that $f(z) = az^n$ for all $z \in \mathbb{E}$. *Hint.* Consider the function $h(z) := \frac{1}{nz^{n-1}} \sum_{j=1}^{n} f(\zeta^j z)$. For the proof of the implication $f(\zeta z) + f(\zeta^2 z) + \cdots + f(\zeta^n z) = naz^n \Rightarrow f(z) = az^n$, verify that the function $k(z) := f(z) - az^n$ satisfies
>
> $$k(\zeta z) + k(\zeta^2 z) + \cdots + k(\zeta^n z) = 0 \quad \text{and} \quad |az^n| + 2\Re(az^n \overline{k(\zeta^j z)}) + \left|k((\zeta^j z)\right|^2 < 1$$
>
> for every $j \in \{0, 1, \ldots, n-1\}$, and consequently $|k(z)|^2 < n(1 - |z|^{2n})$.