Midterm exam **Math 503** **November 10, 2004**

Please do all problems, *in any order.*

1. Create a function which is meromorphic in all of $\mathbb{C}$ which has exactly n poles of order n for each $n \in \mathbb{N}$. This function will have infinitely many poles. You can specify where the poles are. You must show that any series you use converges in an appropriate manner. Give details about any estimates you need.

2. If f is holomorphic in a neighborhood of the closed unit disc, and if $|f(z)| = 1$ when $|z| = 1$, prove that f is a rational function.

Hint One such function is $\dfrac{z-\alpha}{1-\overline{\alpha}z}$ for $\alpha \in D(0,1)$, the open unit disc.

3. Suppose that U is an open subset of $\mathbb{C}$. Prove that there is a sequence of compact subsets of U, $\{K_n\}_{n\in\mathbb{N}}$, so that $\bigcup_{n\in\mathbb{N}} K_n = U$ and $K_n \subseteq \text{interior}(K_{n+1})$ for all $n \in \mathbb{N}$.

4. Prove that the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$ and the punctured unit disc $D(0,1)^* = D(0,1)\backslash\{0\}$ are *not* biholomorphic.

5. Let $\Omega = \{z \in \mathbb{C} : 0 < |z| < \infty\}$. Determine all holomorphic functions f on Ω such that

$$|f(z)| < \frac{1}{|z|^{1/2}} + |z|^{1/2}, \qquad z \in \Omega.$$

Justify your answer.