Midterm exam answers # Math 503 **November 14, 2004**

1. Create a function which is meromorphic in all of $\mathbb{C}$ which has exactly n poles of order n for each $n \in \mathbb{N}$. This function will have infinitely many poles. You can specify where the poles are. You must show that any series you use converges in an appropriate manner. Give details about any estimates you need.

Answer Here's one answer. If j is a positive integer, let $Q_j(z) = \frac{1}{j}\sum_{k=1}^{j} \frac{1}{(z-(j+2^j+ki))^j}$. Then Q_j is a rational function with j poles of order j. If $S_j = D(0,j)$, then $Q_j \in \mathcal{H}(S_j)$ and $\|Q_j\|_{S_j} \leq \frac{1}{2^j}$. Consider now the function $F = \sum_{j=1}^{\infty} Q_j$. Write $F = \sum_{j=1}^{J} Q_j + \sum_{j=J+1}^{\infty} Q_j$ and note that the infinite "tail" is holomorphic on S_J because of Weierstrass's Theorem: each function is holomorphic on S_J and the infinite sum is dominated there by $\sum_{j=J+1}^{\infty} \frac{1}{2^j}$. The M-test shows that the series converges uniformly. The sum is itself holomorphic by Weierstrass's Theorem on uniform convergence of holomorphic functions. On S_J the whole sum for F is meromorphic with j poles of order j for j small enough ($|j+2^j+ji| < J$). Since $\mathbb{C}$ is the union of the S_J's, the function F has the desired properties.

2. If f is holomorphic in a neighborhood of the closed unit disc, and if $|f(z)| = 1$ when $|z| = 1$, prove that f is a rational function.

Hint One such function is $\frac{z-\alpha}{1-\overline{\alpha}z}$ for $\alpha \in D(0,1)$, the open unit disc.

Answer Let $g_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$. So $|g_\alpha(z)| = 1$ when $|z| = 1$, and g_α is rational with no poles in $D(0,1)$ and a simple zero at $\alpha \in D(0,1)$. If f has a zero of order k, a positive integer, at β in $D(0,1)$, then $f(z)/\left(g_\beta(z)\right)^k$ has a removable singularity at β, since the zeros of f (locally, $(z-\beta)^k \cdot$ unit) and $(g_\beta)^k$ (locally, $(z-\beta)^k \cdot$ another unit) cancel. We extend $f(z)/\left(g_\beta(z)\right)^k$ to be holomorphic at β. But f has only a finite number of zeros in $D(0,1)$ since $|f(z)| = 1$ on $\partial D(0,1)$ and $\overline{D(0,1)}$ is compact. Otherwise the zeros would have an accumulation point in $D(0,1)$ which would imply (the Identity Theorem) that f is always 0, contradicting the behavior of f on $\partial D(0,1)$. Each of these zeros has finite multiplicity, so we can repeat the process previously described and obtain $F(z) = \frac{f(z)}{\prod_{j=1}^{n}\left(g_{\beta_j}(z)\right)^{k_j}}$, a function holomorphic in a neighborhood of $\overline{D(0,1)}$ with $|F(z)| = 1$ when $|z| = 1$ and never vanishing in $\overline{D(0,1)}$. The Maximum Modulus Theorem applied to $1/F$ shows that F is constant (with modulus 1) in $D(0,1)$. Thus $f(z) = e^{i\theta}\prod_{j=1}^{n}\left(g_{\beta_j}(z)\right)^{k_j}$, a rational function.

Note Such a function is called a finite Blaschke product.

3. Suppose that U is an open subset of $\mathbb{C}$. Prove that there is a sequence of compact subsets of U, $\{K_n\}_{n\in\mathbb{N}}$, so that $\bigcup_{n\in\mathbb{N}} K_n = U$ and $K_n \subseteq \text{interior}(K_{n+1})$ for all $n \in \mathbb{N}$.

Answer If $U = \mathbb{C}$, then take $K_n = \overline{D(0,n)}$. Otherwise, $\partial U \neq \emptyset$, and for $z \in U$, $d(z) = \inf\{|z-w| : w \in \partial U\} < \infty$. Let $\{z_j\}_{j\in\mathbb{N}}$ be a countable dense subset of U (for example, take the points in U with rational coordinates). Now let K_n be $\cup_{j=1}^{n}\overline{D(z_j,(1-\frac{1}{n+1})d(z_j))}$. Each K_n is a finite union of closed discs and is therefore compact. Since $D(z_j,(1-\frac{1}{n+1})d(z_j)) \supset$

$\overline{D(z_j,(1-\frac{1}{n})d(z_j))}$ we see that $K_n \subseteq \text{interior}(K_{n+1})$. We only need to verify that if $v \in U$ then v is in some K_n, and that will be done if we show that v is in some $D(z_j, d(z_j))$, since that disc is the union of the $D(z_j,(1-\frac{1}{n+1})d(z_j))$'s as $n \to \infty$. Since $d(v) > 0$, there is z_j in $D(v, \frac{1}{3}d(v))$ because of the density of the $\{z_j\}$'s. But $d(z_j) \geq \frac{2}{3}d(v)$. If not, there is $w \in \partial U$ with $|z_j - w| < \frac{2}{3}d(v)$. Since $|v - z_j| < \frac{1}{3}d(v)$ then $|v - w| < d(v)$ which is impossible. Therefore, $v \in D(z_j, \frac{1}{3}d(v)) \subset D(z_j, d(z_j))$.

Comment Another common approach to this problem is write the desired K_n as the intersection of points in U whose distance to ∂U is $\leq \frac{1}{n}$ with $\overline{D(0,n)}$. Of course $U = \mathbb{C}$ is a simple special case. With this approach, there much be still be some verification that K_n is closed, and that $K_n \subseteq \text{interior}(K_{n+1})$. Some use of the triangle inequality is needed.

4. Prove that the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$ and the punctured unit disc $D(0,1)^* = D(0,1)\backslash\{0\}$ are *not* biholomorphic.

Answer Suppose $F : D(0,1)^* \to A$ were such a biholomorphic mapping. The isolated singularity of F at 0 is removable by the Riemann Removable Singularity Theorem, since $|F(z)| < 2$ always. We extend F to $G : D(0,1) \to A$. Since G extends F and F is not constant, G is not constant and holomorphic, and therefore G is open. Let $v = G(0)$. $1 \leq |v| \leq 2$ since $z \mapsto |G(z)|$ is continuous. If $v \in A$ then there is $w \in D(0,1)^*$ with $G(w) = G(0)$. But some neighborhood of w and some neighborhood of 0 are both mapped onto a neighborhood of v since G is open. Thus there is $\tilde{v}$ in the neighborhood of v so that $G^{-1}(\tilde{v}) = F^{-1}(\tilde{v})$ is at least two points in $D(0,1)^*$, which is impossible. If now $v \in \partial A$, then since G is open at 0, a neighborhood of 0 is mapped onto a neighborhood of v. This means some point of $D(0,1)^*$ is mapped by F outside of A which is impossible. Therefore there is no mapping F.

5. Let $\Omega = \{z \in \mathbb{C} : 0 < |z| < \infty\}$. Determine all holomorphic functions f on Ω such that

$$|f(z)| < \frac{1}{|z|^{1/2}} + |z|^{1/2}, z \in \Omega.$$

Justify your answer.

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Answer f has a Laurent series in Ω: $f(z) = \sum_{n=-\infty}^{n=\infty} a_n z^n$ where $a_n = \frac{1}{2\pi i}\int_{\partial D(0,R)} \frac{f(z)}{z^{n+1}}\,dz$ for $R > 0$. Thus $|a_n| \leq \frac{M(f,R)}{R^n}$ by the ML inequality, where $M(f,R) = \sup\{|f(z)| : |z| = R\}$. Now $M(f,R) \leq R^{-1/2} + R^{1/2}$ so that if $n > 0$ and $R > 1$, $|a_n| \leq \frac{(1+R^{1/2})}{R^n}$. Since $n > 0$ and is an integer, $\lim\limits_{R\to\infty} \frac{(1+R^{1/2})}{R^n} = 0$. Thus $a_n = 0$ for such n. If now $n < 0$ and $0 < R < 1$, $|a_n| \leq \frac{(R^{-1/2}+1)}{R^n}$. Since $n < 0$ and is an integer, $\lim\limits_{R\to\infty} \frac{(R^{-1/2}+1)}{R^n} = 0$. Thus $a_n = 0$ for such n. f must therefore be constant, a_0. Further, the constant value has modulus less than the minimum value of $\sqrt{x} + \frac{1}{\sqrt{x}}$ on $(0,\infty)$. This minimum value occurs at $x = 1$ and is 2.