Consider the linear automorphism of $\mathbb{C}^2$: $GL_2(\mathbb{C})$.

This is $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ab - cd \neq 0$, & $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} a z_0 + b z_1 \\ c z_0 + d z_1 \end{pmatrix}$. Now M takes lines through 0 to lines through 0 (i.e., if $z \sim \tilde{z}$, then $Mz \sim M\tilde{z}$ since M is linear). And M is a bijection on such lines. So M acts on equivalence classes of lines.

$[M]([z_0 : z_1]) = [a z_0 + b z_1 : c z_0 + d z_1]$. $[M]$ is a bijection (indeed, a homeomorphism!) of $\mathbb{CP}^1$.

What does it look like in $\mathbb{C}$? i.e., what is $[M](\pi \circ \sigma(z))$ (Note most of the time people omit all the maps and parentheses!) Let's "compute":

$[M](\pi \circ \sigma(z)) = [M]([z:1]) = [az+b : cz+d]$

$= [\frac{az+b}{cz+d} : 1] = \pi \circ \sigma\left(\frac{az+b}{cz+d}\right)$. So $[M]$ is

if $cz+d \neq 0$

the same as the mapping $z \rightsquigarrow \frac{az+b}{cz+d}$ of $\mathbb{C}$, away from where $cz+d=0$, which is mapped to ∞.

Note that $\infty = [1:0]$ is mapped by $[M]$ to $[a:c]$.

If $c=0$, $[M](\infty) = \infty$. If $c \neq 0$, ∞ is sent to $\pi \circ \sigma\left(\frac{a}{c}\right)$.