Projective space

Most pretentious approach:

Let V be a vector space over a field F. Then we put P(V) to be the set of all 1 dimensional subspaces of V (the lines). In particular, V can be chosen to be $L^2(\mathbb{R})$(important in Quantum Mechanics, among others), or a finite dimensional vector space F^n(important in Combinatorics, for example).

This approach is maybe too difficult to achieve our **aim**: the study of $\mathbb{C}P^1$ - the 1-dimensional complex projective space. So, we will define it first as an equivalence relation, in the following way:

$$\begin{aligned}
\mathbb{C}^2 &= \{(\alpha,\beta) : \alpha,\beta \in \mathbb{C}\} \\
\mathbb{C}^{2*} &= \mathbb{C}^2 - \{(0,0)\} \\
(\alpha_1,\beta_1) \sim (\alpha_2,\beta_2) &\Leftrightarrow \exists \lambda \in \mathbb{C}^* : \lambda(\alpha_2,\beta_2) = (\alpha_1,\beta_1) \\
\mathbb{C}P^1 &= \mathbb{C}^{2*}/\sim
\end{aligned}$$

The equivalence classes in $\mathbb{C}P^1$ are the one dimensional subspaces of $\mathbb{C}^2$ over $\mathbb{C}$.

How to define a topology on $\mathbb{C}P^1$?

Suppose $(\alpha,\beta) \in \mathbb{C}^{2*} \Rightarrow [(\alpha,\beta)] \in \mathbb{C}P^1 \Rightarrow [\alpha,\beta] = [\lambda(\alpha,\beta)] \in \mathbb{C}P^1$. A unique representative (in some sense) of the class $[\alpha,\beta]$ can be described in the following way:

$$[\alpha,\beta] = \begin{cases} [\frac{\alpha}{\beta},1] & \text{if } \beta \neq 0 \longleftrightarrow [z,1], z \in \mathbb{C} \\ [1,0] & \text{if } \beta = 0 \end{cases}$$

So, we can imagine $\mathbb{C}P^1$ as a copy of $\mathbb{C}$ together with a distinct element $[1,0]$, which intuitively will be ∞.

For $z \neq 0$, we also have $[z,1] = [1,\frac{1}{z}]$. We can use this to define a topology on $\mathbb{C}P^1$. For $z \in \mathbb{C}^*$, we make the identification:

$$z \longleftrightarrow [z,1] \overset{\sharp}{\longleftrightarrow} [1,z] \longleftrightarrow \frac{1}{z}$$

This is a continuous overlap mapping from the open sets $\mathbb{C}^*$ to $\mathbb{C}^*$. Therefore, if we put $\mathbb{C}_z = \mathbb{C} \cup \{[1,0]\}$ and $\mathbb{C}_w = \mathbb{C} \cup \{[0,1]\}$ and consider the above mentioned corespondence between $(\mathbb{C}_z \cup \mathbb{C}_w)/\sharp \longleftrightarrow \mathbb{C}P^1$ we maybe can view $\mathbb{C}P^1$ to be homeomorphic to the one point compactification of $\mathbb{C}$. That is $\mathbb{C}P^1 \cong \mathbb{C} \cup \{\infty\}$, where the neighborhoods of ∞ are of the form $\{z \in \mathbb{C} : |z| > A, A \in \mathbb{R}\}$.

Another way to put a topology on $\mathbb{C}P^1$ is to consider the projection:

$$\mathbb{C}^{2*} \xrightarrow{\pi} \mathbb{C}^{2*}/\sim \cong \mathbb{C}P^1$$

and put a topology on $\mathbb{C}P^1$ such that π is continuous. In general, it is taken the strongest topology in which the mapping π remains continuous. This topology is equivalent to the one just defined.

Yet another equivalent way is to consider the stereographic projection of the Riemann sphere. This would make $\mathbb{C}P^1 \cong S^2$.

We consider the following diagram:

$$(\alpha, \beta) \in \mathbb{C}^{2*} \to \mathbb{C}P^1 \xrightarrow{F} \mathbb{C}P^1 \to \mathbb{C}^2 \ni (f(\alpha,\beta), g(\alpha,\beta))$$

We would want:

$$\begin{aligned}
(\alpha_1,\beta_1) \sim (\alpha_2,\beta_2) &\Rightarrow (f(\alpha_1,\beta_1), g(\alpha_1,\beta_1)) \sim (f(\alpha_2,\beta_2), g(\alpha_2,\beta_2)) \Leftrightarrow \\
(\alpha_1,\beta_1) = \lambda(\alpha_2,\beta_2) &\Rightarrow (f(\alpha_1,\beta_1), g(\alpha_1,\beta_1)) = \mu(f(\alpha_2,\beta_2), g(\alpha_2,\beta_2)) \Rightarrow \\
f(\lambda(\alpha_2,\beta_2)) &= \mu f(\alpha_2,\beta_2) \text{ and } g(\lambda(\alpha_2,\beta_2)) = \mu g(\alpha_2,\beta_2)
\end{aligned}$$

This leads us to considering **homogeneous polynomials**: $P \in \mathbb{C}[z,w]$ is homogeneous iff $\exists n \in \mathbb{N}$ such that $\forall \lambda, z, w \in \mathbb{C} : P(\lambda z, \lambda w) = \lambda^n P(z,w)$. In this case, we say that P is a homogeneous polynomial of degree n.

Example for n=3: $P(z,w) = Az^3 + Bz^2w + Czw^2 + Dw^3$.

If we return to the diagram we have considered, we may choose F to be $F([z,1]) = [\frac{P_1(z,1)}{P_2(z,1)}, 1]$, where P_1, P_2 are homegenous polynomials of the same degree n. That is, we may consider the mappings $\frac{A}{B}$, where $A, B \in \mathbb{C}[t]$ have the same degree n.

Now, our aim is to make F be a holomorphic mapping, in some sense. For this, we need to prepare the setting in which we work, that is we want to view $\mathbb{C}P^1$ as a **Riemann surface** (not a Riemann manifold). We say that X is a n-dimensional topological manifold if X is a topological space locally homeomorhic to $\mathbb{R}^n$. Usually we want to make our life easier so we will require that the manifold satisfies some additional properties as: it is Hausdorff, connected, σ - compact (X can be written as an ascending union of compact sets, which will allow us to consider only a countable family of charts), paracompact.

We made several observartions as for instance that a connected space is not necessarily Hausdorff. For this we considered the real line from which we deleted 0 and replaced it by 2 points. The topology changes in that the neighborhoods of the 2 additional points become the neighborhoods of 0 from which we delete zero and add the appropriate point. Such a space, remains connected, but it is not Hausdorff because the 2 additional points cannot be separated by disjoint neighborhoods. There was another example about paracompact spaces, but I didn't understand it.

Suppose that (U, φ) and (V, ψ) are 2 overlapping charts in a 2-dimensional manifold. If $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap)$ is holomorphic for any 2 overlapping coordinate charts (U, φ) and (V, ψ), then X is called a **Riemann surface**.

We say that a continuous mapping f between 2 Riemann surfaces X and Y is holomorphic if no matter how we choose a point $p \in X$ and a chart (U, φ) around p and (V, ψ) around $f(p)$, then $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is holomorphic.

An important, amazingly 'simple' result that we mentioned is the **Uniformization theorem:** If X is a simply connected Riemann surface then X is biholomorphic to $D(0,1), \mathbb{C}$ or $\mathbb{C}P^1$.

Now, we come back to $\mathbb{C}P^1$ viewed as S^2 and cover it by 2 chats - the projection from the North pole $(0,0,1)$ and from the South pole $(0,0,-1)$. This is a way to define $\mathbb{C}P^1$ as a Riemann surface.

We want to determine Aut$(\mathbb{C}P^1) = \{f : \mathbb{C}P^1 \to \mathbb{C}P^1 : f$ is 1-1, onto and holomorphic $\}$. We look at the f's which stabilizes $\infty \Rightarrow f(\infty) = \infty \Rightarrow$ f restricted to $\mathbb{C}_z$ is a proper holomorphic mapping. Because f is also bijective, from what we have proved before (we remember Aut$(\mathbb{C})$), it follows that f has the form $f(z) = az + b, a \neq 0$. Now we consider the transitive part of Aut$(\mathbb{C}P^1)$, that is the f's with the property $f(\infty) = z_0 \in \mathbb{C}$. If we compose such an f with $\varphi(z) = \frac{1}{z - z_0}$ $(z_0 \stackrel{\varphi}{\to} \infty)$, we obtain a mapping which fixes ∞ (a mapping from the stabilizer of Aut$(\mathbb{C}P^1)$). This helps us to show that f has the form $f(z) = \frac{az+b}{cz+d}$, where $(a, b), (c, d)$ are linearly independent.

What we have shown (almost):

$$Aut(\mathbb{C}P^1) = \left\{ \frac{az+b}{cz+d} : det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \right\}$$

It can be shown that Aut$(\mathbb{C}P^1)$ is a group and its elements are called liniar fractional transformations, Möbius transformations, etc. It is also denoted $PGL_2((C)$ and contains $SO(3)$(rotations of the unit sphere), $SU(1,1)$ (automorphisms of the unit disc), Aff$(\mathbb{C}) = \{az + b : a \neq 0\}$(automorphisms of the complex plane).