Math 503 **Final Exam** **12/17–19/2007**

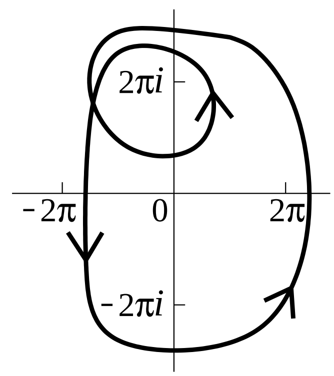
Please do all problems.

1. Suppose that u and v are the real and imaginary parts respectively of an entire function f. Find all f such that $u = v^2$.

2. Suppose U is an open subset of $\mathbb{C}$. Show that there is a sequence of <u>compact subsets of U</u>, $\{K_n\}_{n\in\mathbb{N}}$ so that $\bigcup_{n=1}^{\infty} K_n = U$ with each K_n <u>contained in the interior</u> of K_{n+1}.

3. Suppose that f is an entire function, and that for all $|z| > 7$, $|f(z)| \leq 5e^{(|z|^2)}$. Find explicit positive numbers A and B so that if $|z| > A$, then $|f''(z)| \leq Be^{(3|z|^2)}$

4. Suppose $f(z) = \dfrac{1}{z^2\,(e^z - 1)}$.



a) Find and classify (removable, pole, essential) *all* isolated singularities of f. If the isolated singularity is a pole, tell the order of the pole and the residue of f at the pole.

b) Compute $\int_\sigma f(z)dz$ where σ is the closed curve displayed to the right.

5. What is the automorphism group of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$? That is, describe all biholomorphic mappings from $\mathbb{C}^*$ to itself. Decide whether or not this group is transitive. If it is not, describe the orbit and stabilizer of 1, and also describe the orbit and stabilizer of 2.

Note The *orbit* of a point under a collection of mappings is the set of image points. The *stabilizer* (also called the *stabilizer subgroup* or the *isotropy subgroup*) of a group of bijections is the subgroup which keeps the indicated object (here, a point) fixed.

6. Suppose f is an entire function, and that, for all $z \in \mathbb{C}$, $f(z + i) = f(z + 1) = f(z)$ (so f is *doubly periodic* with periods i and 1). Prove that f is constant.