Math 503 **Midterm Exam** **11/13/2007**

Please do all problems.

1. Suppose that σ is a permutation of the positive integers, $\mathbb{N}$ (so $\sigma : \mathbb{N} \to N$ is a bijection, that is, one-to-one and onto), and that $\sum_{j=1}^{\infty} a_j$ converges absolutely and its sum is S.

Prove that $\sum_{j=1}^{\infty} a_{\sigma(j)}$ converges, and that the sum of this series is S.

2. Suppose that $f(z) = \sum_{n=1}^{\infty} \left(\frac{1}{n!}\right)\left(\frac{1}{z-n}\right)$.

Prove that the sum converges absolutely for all $z \in \mathbb{C} \setminus \mathbb{N}$, where $\mathbb{N}$ is the collection of positive integers. Also prove that the sum converges uniformly on all compact subsets of $\mathbb{C} \setminus \mathbb{N}$ and that the sum is meromorphic in $\mathbb{C}$. Compute the residue of f at $n \in \mathbb{N}$, explaining why your answer is correct.

Hint In any disc centered at 0, write the sum defining $f(z)$ as an initial sum plus an infinite tail, and then analyze the infinite tail.

3. Suppose W is a discrete closed subset of a connected open subset, U, of $\mathbb{C}$. Prove that W is countable (it may be finite or infinite, but it is countable!) and that $U \setminus W$ is connected.

4. Describe all uniformly continuous entire functions (with proof).

5. Suppose $U^* = D_1(0)\backslash\{0\}$ (the unit disc with 0 deleted) and A is the annulus of all $z \in \mathbb{C}$ with $1 < |z| < 2$. Prove that U^* and A are *not* biholomorphic: there is no 1–to–1 and onto mapping between these sets so that the map and its inverse are both holomorphic.