Math 503 # Mittag-Leffler & Weierstrass 12/3/2007

There are almost magical techniques for proving the existence of meromorphic and holomorphic functions in $\mathbb{C}$ with rigidly specified behavior (poles and zeros). The principal results are called the Mittag-Leffler Theorem and the Weierstrass Factorization Theorem. Here are some direct proofs of both results. A few consequences are mentioned.

Mittag-Leffler Theorem Suppose W is a closed, discrete subset of $\mathbb{C}$, and suppose that for each $w \in W$, a polynomial P_w in $\mathbb{C}[z]$ with no constant term is selected. Then there is a meromorphic function f defined in $\mathbb{C}$ whose set of poles is W such that the principal part of f at each $w \in W$ (this is the sum of the terms of negative degree in the Laurent series for f at w) is $P_w\left(\frac{1}{z-w}\right)$.

Proof If W is finite, then f satisfying the theorem is the sum of the principal parts, a rational function. Otherwise we first write W as a disjoint union of sets W_n where n is a non-negative integer. W_0 will be those w's in W with $|w| \leq 1$, while, more generally, if n is at least 1, W_n is the collection of w's in W with $n < |w| \leq n+1$. Note that since W is discrete and closed, each of the W_n's has at most finitely many elements. Of course, some of them may be empty. Then define Q_n for $n \geq 0$ by $Q_n(z) = \sum_{w \in W_n} P_w\left(\frac{1}{z-w}\right)$ (this is a finite sum!). If W_n is empty, then $Q_n = 0$. These are the sum of the principle parts, the pieces of the singularities, in each of the annular regions between an integer and its successor (that integer $+1$).

Now consider W_n for $n \geq 1$. All w's in this W_n must have $|w| \geq n$ so that the sum defining Q_n is holomorphic in some disc of radius r, where $r > n$. This is because W_n is finite, each w of W_n has modulus greater than n, and a minimum of a finite set is one of the set's elements (a specific $|w|$ with $|w| = r > n$). Since Q_n is holomorphic in $D_n(0)$, it is equal to a power series centered at 0 valid in all of that disc. The series will converge uniformly on compact subsets. Therefore there is a partial sum V_n of this power series (just a polynomial!) so that if $|z| \leq n$ then $|Q_n(z) - V_n(z)| < \frac{1}{2^n}$.

Here is a "recipe" for f: $f(z) = Q_0(z) + \sum_{n=1}^{\infty}(Q_n(z) - V_n(z))$. Some verification is necessary.

• We will show that if z is *not* in W, the sum defining f converges absolutely. We know there is an integer N with $|z| \leq N$. Break up the sum defining $f(z)$ in two parts, $f_{\text{in}}(z) + f_{\text{out}}(z)$, where $f_{\text{in}}(z) = Q_0(z) + \sum_{n=1}^{N}(Q_n(z) - V_n(z))$ and $f_{\text{out}}(z) = \sum_{n=N+1}^{\infty}(Q_n(z) - V_n(z))$

The series defining f_{out} (the infinite series, and the sum for f_{in} is finite) converges absolutely and uniformly for all z with $|z| \leq N$. This follows because if $n > N \geq |z|$, then $|Q_n(z) - V_n(z)| < \frac{10}{2^n}$. Use the Weierstrass M-test.

• We also now observe that given any disc centered at 0 in $\mathbb{C}$, the series defining f can be written as a sum of a rational function plus a function which is holomorphic in that disc. Additionally the rational function has the desired principal parts for all w's in W which happen to be in the disc. This of course also follows from the $f_{\text{in}} + f_{\text{out}}$ decomposition described previously. Surely f_{out} is holomorphic in the disc, since it is the sum of a uniformly convergent series of functions holomorphic in the disc. f_{in} is a *finite* linear combination of inverse powers of $z - w$ (for those w's in W which are in the disc) and of polynomials in z. That's certainly a rational function! And, of course, the rational function does have the desired principal parts. ■

Review of the proof The first paragraph "dissects" or breaks up W into parts which are each finite and nicely chosen geometrically so consequences of the organization lead to simple estimates. The second paragraph defines the polynomials which will control the principal parts in the appropriate discs. Then f's recipe is given, and the next two paragraphs verify the properties of the recipe.

First comment I was wrong in class when I asserted that the "bookkeeping" involved in such a proof would be forbidding. I was thinking algorithmically. I wanted a more definite "formula" or "procedure" for the function f, that is, for selecting the polynomials which are involved with its description. Certainly if W is a sequence of w's whose growth (rate of increase of $|w|$) is known, we can use geometric series arguments effectively to get the polynomials needed to balance the principal parts. But the proof above avoids that consideration. There is no "effective formula" given, but the phrases "will converge uniformly" (applied to the power series for a holomorphic function) and "we can choose a partial sum ... so that" applied to the same series are the existential (?) version of effectively estimating the geometric series remainders.

Second comment Be aware that the theorem does *not* assert that $\sum_{w \in W} P_w(z)$ converges or that $\sum_{n=1}^{\infty} Q_n(z)$ converges! That might be true (someone might have put in $\frac{1}{n!}$ factors, after all). The convergence of the series defining f is delicate. We have altered the terms subtly so that the series for f does converge in the way we would like.

Third comment A version of Mittag-Leffler is true for any open subset of $\mathbb{C}$, as I declared in class. But a proof needs some topological "dissection" of the set as a nice increasing union of compact sets. Some further knowledge of analysis is very useful also, since we need the analog of partial sums of Taylor series. A result called Runge's Theorem (also classical) provides such approximations. But I won't prove Runge's Theorem in this course.

Now let's consider the zeros of an entire function, f. The exponential function has *no zeros* (any entire function with no zeros can be written as e^g where g is entire since $\mathbb{C}$ is simply connected). Suppose that f does has zeros or roots. If f has a finite number of roots, then f can be written as a product of a polynomial and $e^{g(z)}$.

From here on we suppose that f has infinitely many roots. Of course, one such function is the zero function . f will be that function if the set of roots has an accumulation point. Let's also assume that f is not constant. Then the set of roots is countable with no accumulation point (a closed, discrete subset of $\mathbb{C}$). But we need to think also about multiplicity. In complex analysis and algebra z^3 naturally seems like it has 3 roots which all happen to be 0. So we will count the multiplicity of the roots, also. Then each complex number may occur as a root many times, but only finitely many times. Indeed, if $\{z_n\}$ is the sequence of roots, then we know these **Necessary Root Facts**:

1. For any w in $\mathbb{C}$, the equation $z_n = w$ is true for only finitely many n's.
2. For any real A, the inequality $|z_n| < A$ is true for only finitely many n's.

In what follows, we'll assume that these two properties are true. The zero set of a non-constant entire function with infinitely many roots has both of these properties, and we will show that these simple necessary conditions are *sufficient* for the existence of a non-constant entire f with the sequence $\{z_n\}$ as its zeros, with multiplicities.

If we had only a finite number of roots, then we could easily write a product which had the desired roots. Suppose we try a straightforward approach: take the sequence $\{z_n\}$ and write $\prod_{n=1}^{\infty}(z - z_n)$ and this is very nice except *what does it mean?*

Infinite products

We might want to declare that an infinite product converges if the sequence of partial products (analogous to partial sums) converges. Consider the following examples:

$\prod_{n=0}^{\infty} n$. (This product starts with $n = 0$.) This candidate for an infinite product is 0 even though the terms are unbounded. The first term, 0, "kills" all of the partial products, which are the products of an initial segment of this infinite product.

$\prod_{n=1}^{\infty} \frac{1}{n}$. The partial products clearly approach 0, so the infinite product, if simply defined as a limit of partial products, will be 0. This infinite product is 0 even though none of its terms are 0.

Consideration of such examples has led to the widespread adoption of a more careful definition of a convergent infinite product. If there's no more intricate definition than the limit of partial products, inserting a 0 anywhere in the product would imply that the product would "inherit" none of the properties of its factors.

Definition of convergent infinite product

The infinite product $\prod_{n=1}^{\infty} w_n$ (where the w_n's are all supposed to be complex numbers) is said to converge if there is an integer N so that for all $n \geq N$, w_n is *not* 0, and if also $\lim_{k\to\infty} \prod_{n=N}^{k} w_n$ exists and is *not* 0.

With this definition, the individual terms of a convergent infinite product must tend to 1. It makes sense then to change statements about infinite products from $\prod_{n=1}^{\infty} w_n$ to something like $\prod_{n=1}^{\infty}(1 + a_n)$. We will investigate how $a_n \to 0$ and when the infinite product converges as a consequence.

Proposition Suppose that $\{a_n\}$ is a sequence of complex numbers, and that $\sum_{n=1}^{\infty} |a_n|$ is finite. Then the infinite products $\prod_{n=1}^{\infty}(1 + |a_n|)$ and $\prod_{n=1}^{\infty}(1 + a_n)$ both converge.

Proof If $\sum_{n=1}^{\infty} |a_n|$ converges, then eventually the terms will be less than $\frac{1}{2}$, so let me assume that they are all less than $\frac{1}{2}$. But look at log near 1. If $|z| < \frac{1}{2}$, then $\log(1+z) = z + \text{ERR}(z)$, where $|\text{ERR}(z)| < 2|z|^2$ (look at the Taylor series and overestimate the tail – this is done later with more details). Then $\log\left(\prod_{n=1}^{\infty} 1+a_n\right) = \left(\sum_{n=1}^{\infty} a_n\right) + \text{ERROR}$ where now ERROR is the sum of the possible errors. Remember in this series the terms all have modulus less than $\frac{1}{2}$ (possibly a tail of the original series) so that squaring makes terms even smaller. Therefore the series of errors which make up ERROR must converge. So since log of the product converges, the product must also. ■

Comment The principal hypothesis of this proposition implies that the product's convergence is very much like the convergence of an absolutely convergent *series*. So, in fact, $\left|\prod_{n=1}^{\infty}(1+a_n)\right| \leq \prod_{n=1}^{\infty}(1+|a_n|)$ is true. Also rearrangement is allowed (similar to what happens to sums of absolutely convergent series): for infinite products which satisfy this hypothesis, we can rearrange the factors, the result will always converge, and the value of the permuted product will be the same. This stability is very useful. All of the infinite products following this note will "converge absolutely" in the sense of this result.

Proposition If $\{g_n\}$ is a sequence of continuous complex-valued functions on $S \subset \mathbb{C}$, and $\sum_{n=1}^{\infty} |g_n(z)|$ converges uniformly on S, then $\prod_{n=1}^{\infty}(1+|g_n(z)|)$ and $\prod_{n=1}^{\infty}(1+g_n(z))$ both converge and their values are continuous functions on S.

We can say a bit more for holomorphic functions.

Theorem Suppose U is open and connected in $\mathbb{C}$, $\{g_n\}$ is a sequence of holomorphic functions in U, none of which are constant, and suppose that $\sum_{n=1}^{\infty} |g_n(z)|$ converges uniformly on compact subsets of U. Then $F(z) = \prod_{n=1}^{\infty}(1+g_n(z))$ converges for all z in U and is holomorphic in U. F is not identically 0, and if $F(w)=0$, then the order of the zero of F at w is equal to the sum of the orders of the zeros of $1+g_n$ at w.

I won't prove this, but let's talk about it. Suppose z is in U. Since the product for $F(z)$ converges, there is an N so that $\prod_{n=N}^{\infty}(1+g_n(z))$ converges and is non-zero. If $F(z)=0$, only finitely many of the $(1+g_n)$'s for $n<N$ can be 0 at z, and (the functions aren't constant) these zeros have finite order. So the sum of the orders of the zeros of the $(1+g_n)$'s must be finite. Then the theorem follows from the previous result and the fact that uniformly convergent sequences of holomorphic functions have holomorphic limits.

Now we create f. We have a sequence $\{z_n\}$ satisfying the **Necessary Root Facts**. We initially tried $\prod_{n=1}^{\infty}(z-z_n)$ but now rewrite the infinite product as $\prod_{n=1}^{\infty}\left(1-\left(\frac{z}{z_n}\right)\right)$. If some of the z_n's are 0, I will just multiply the result by z^K so I will assume all of the z_n's to be considered aren't 0. We will generally not be lucky enough to have $\sum_{n=1}^{\infty}|z/z_n|$ converge (z_n could just be n, for example). We will need convergence producing factors. The factors should be $e^{\text{SOMETHING}}$ since we don't want to create more zeros of the resulting function.

Now look at $\prod_{n=1}^{\infty}\left(1-\left(\frac{z}{z_n}\right)\right)e^{h_n(z)}$. Consider the log of this product. (Given z in $\mathbb{C}$, everything I write is true for sufficiently large n uniformly in a neighborhood of z since $\lim_{n\to\infty}|z_n| = \infty$.) So we have a series whose n^{th} term is $\log\left(1-\left(\frac{z}{z_n}\right)\right) + h_n(z)$. If I can get h_n's so this is $O\left(\frac{1}{2^n}\right)$ as $n\to\infty$ for all z's in some disc around 0, then we'd be done.

Fix some positive integer n, and consider the z's which satisfy $|z| < \frac{|z_n|}{2}$. Then we know that $\log\left(1-\left(\frac{z}{z_n}\right)\right)$ can be written using a convergent power series: $\log(1+w) = \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}w^j$ for $|w|<1$. We do some calculus to get a partial sum and an error term: $\frac{1}{1+w} = \sum_{j=0}^{\infty}(-1)^j w^j = \sum_{j=0}^{m-1}(-1)^j w^j + \sum_{j=m}^{\infty}(-1)^j w^j = \sum_{j=0}^{m-1}(-1)^j w^j + \frac{(-1)^m w^m}{1+w}$. Suppose that $|w|<\frac{1}{2}$ and integrate along the line segment connecting 0 and w. We get $\sum_{j=1}^{m}\frac{(-1)^{j+1}}{j}w^j + \int_0^w \frac{(-1)^m v^m}{1+v}dv$. We estimate the second term, the error, with ML. Surely $|\text{top}|\cdot|\text{length}| \le |w|^{m+1}$, and, as for the bottom, $|1+v| \ge 1-|v| > \frac{1}{2}$ since $|w|<\frac{1}{2}$. So *if* we know that $|w|<\frac{1}{2}$, then $\log(1+w) = \sum_{j=1}^{m}\frac{(-1)^{j+1}}{j}w^j + \text{ERROR}_m(w)$ where $|\text{ERROR}_m(w)| < 2|w|^{m+1}$. Now "plug in" $-\frac{z}{z_n}$ for w to get $\log\left(1-\frac{z}{z_n}\right) = \sum_{j=1}^{m}\frac{(-1)^{j+1}}{j}\left(-\frac{z}{z_n}\right)^j + \text{ERROR}_m\left(-\frac{z}{z_n}\right)$ where $\left|\text{ERROR}_m\left(-\frac{z}{z_n}\right)\right| < 2\left|-\frac{z}{z_n}\right|^{m+1}$ if $\left|\frac{z}{z_n}\right| < \frac{1}{2}$.

Now simplifications occur. For example, all of the minus signs except one cancel. Since we are considering only the z's where $\left|\frac{z}{z_n}\right| < \frac{1}{2}$, we can "forget" the hypothesis. The error estimate, again because $\left|\frac{z}{z_n}\right| < \frac{1}{2}$, is easier: $\log\left(1-\frac{z}{z_n}\right) = -\sum_{j=1}^{m}\frac{\left(\frac{z}{z_n}\right)^j}{j} + \text{ERROR}_m\left(-\frac{z}{z_n}\right)$ where $\left|\text{ERROR}_m\left(-\frac{z}{z_n}\right)\right| < \frac{1}{2^m}$ for our z's.

I now know *how to select the correct* h_n! Take m so that $\left|\frac{z}{z_n}\right|^m < \left(\frac{1}{2}\right)^n$ (n and a non-zero z_n are given, and we can take such an m which I will call m_n). Let $h_{m_n}(z) = \sum_{j=1}^{m_n}\frac{\left(\frac{z}{z_n}\right)^j}{j}$ (the minus sign has been dropped since we want the difference to be small).

Weierstrass Factorization Theorem If a sequence $\{z_n\}$ satisfies the **Necessary Root Facts**, there is an entire function f whose zero set counting multiplicities is that sequence. f is a specific infinite product as described above, with a possible z^K factor if needed.

The h_n's described here are involved in the definition of the *Weierstrass elementary factors* and the result above, suitably formulated, is usually called the *Weierstrass Factorization Theorem*. There's also a similar result for any open subset of $\mathbb{C}$. But now a few fantastic results which follow from this theorem.

First, an "algebraic" result I mentioned a while ago, proved for $\mathbb{C}$: the quotient field of the ring of *holomorphic functions* on $\mathbb{C}$ is the set of *meromorphic functions* on $\mathbb{C}$:

Quotient Field Theorem Any meromorphic function defined on $\mathbb{C}$ is a quotient of holomorphic functions (entire functions) on $\mathbb{C}$.

Proof Suppose F is a meromorphic function and suppose that P is its pole set, counted with multiplicities. So if w is a pole of order N, then w appears N times in the sequence $\{z_n\}$. Now create a holomorphic function g whose zero set is *exactly* P. If w is an element of the pole set of F, then we know local descriptions of F and g near w: so $g(z) = (z-w)^N h_1(z)$ and $F(z) = (z-w)^{-N} h_2(z)$ near w, where h_1 and h_2 are holomorphic near w and their values at w are non-zero (they are local multiplicative units). Thus locally, $(g \cdot F)(z) = h_1(z)h_2(z)$ is a unit near w: a non-zero holomorphic function (more precisely, the product $g \cdot F$ has a removable singularity at w, which we will think of as "removed"). Notice that away from the pole set of F, the product $g \cdot F$ is also holomorphic. Therefore this product defines a function f which is holomorphic in $\mathbb{C}$. We get $F = \frac{f}{g}$. ■

There's an amazing result on interpolation which uses both the Weierstrass Factorization Theorem (above) and the Mittag-Leffler Theorem.

Interpolation Theorem Suppose $\{z_n\}$ is a closed discrete sequence in $\mathbb{C}$, and $\{w_n\}$ is *any* sequence in $\mathbb{C}$. Then there is an entire function f so that for all n, $f(z_n) = w_n$.

Proof Use the Weierstrass Factorization Theorem to create a function whose zero set is $\{z_n\}$. Just as before, the local picture of this function near a point z_n in its zero set is $(z - z_n)h(z)$, and $h(z_n)$ is non-zero. Then use the Mittag-Leffler Theorem to create a meromorphic function with simple poles at each z_n, and with principal part at z_n equal to $\left(\frac{w_n}{h(z_n)}\right)\left(\frac{1}{z-z_n}\right)$. The product of these two functions has the desired values. ■

Actually you can do much much better: you can specify arbitrarily a finite initial "chunk" of the Taylor series of a holomorphic function at any closed discrete sequence of points in $\mathbb{C}$ (just use Mittag-Leffler with higher order singularities and Weierstrass with higher order zeros). Compare this result with the problem in the 0^{th} homework assignment, which asserted that a power series can grow arbitrarily fast on the integers. We can now prove something much more precise.

The classical literature is full of very precise descriptions of factorizations for specific functions, such as $\sin(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ which Euler believed. This *is* a convergent infinite product since $\sum_{n=1}^{\infty} \frac{z^z}{n^2}$ converges absolutely and locally uniformly. Since the infinite product has the same zeros as sine, the quotient is a non-vanishing entire function. More work must be done to verify that the quotient is actually 1.

The proofs given here use very classical techniques, and can be extended, with some effort, to other contexts, such more than one complex variable. There are other methods to "construct" holomorphic functions, connected with partial differential equations and with algebraic geometry, which give beautiful insight and can be used in many situations.

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