Math 503 **Problem Set 0** **9/4/2007**

Solutions are due at the beginning of class on Friday, September 7, 2007.

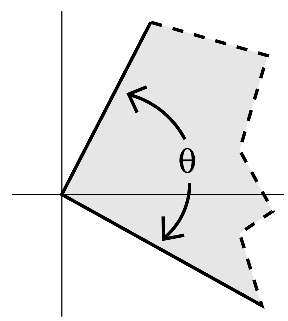
1. In this problem, A is a non-empty subset of $\mathbb{R}$. A real-valued function g defined on A is *bounded* if $\|g\|_\infty = \sup\limits_{x \in A} |g(x)| < \infty$. Also in this problem, $\{f_n\}_{n \geq 1}$ is a sequence of bounded real-valued functions defined on A with $b_n = \|f_n\|_\infty$.

a) Suppose that $\{f_n\}$ converges uniformly on A to a function, f. Prove that f is bounded, and, if $b = \|f\|_\infty$, $b = \lim\limits_{n \to \infty} b_n$.

b) Suppose that $\{f_n\}$ converges pointwise on A to a function, f. Must f be bounded? Either prove that f is bounded, or give a counterexample.

c) Suppose that $\{f_n\}$ converges pointwise on A to a bounded function, f, and that $b = \|f\|_\infty$. Must $b = \lim\limits_{n \to \infty} b_n$? Either prove that this equality is correct, or give a counterexample.

2. Part of an open cone with vertex the origin and aperture or opening θ is shown in the diagram to the right. Give a precise definition (in either $\mathbb{R}^2$ or $\mathbb{C}$) of such an object (it is open, so only the inside of the cone should be included!). Fix θ in $(0, \pi)$, and suppose W is an open cone with aperture θ. Prove the following *reverse triangle inequality* (using algebraic manipulation in either $\mathbb{R}^2$ or $\mathbb{C}$):



There is $C_\theta > 0$ depending on θ (and not on W) so that if z_1 and z_2 are in W, then $C_\theta\left(|z_1| + |z_2|\right) \leq |z_1 + z_2|$.

3. Suppose $\{t_n\}_{n \geq 1}$ is a sequence of positive real numbers. Prove that there is a power series $\sum\limits_{n=0}^{\infty} a_n x^n = S(x)$ which converges for all $x \in \mathbb{R}$ so that $S(n) \geq t_n$ for all $n \in \mathbb{N}$.

Hint Choose $\alpha_n > 0$ and $p_n \in \mathbb{N}$ so that the monomial $\alpha_n \left(\frac{x}{n}\right)^{p_n}$ is greater than t_n when $x = n$ and is less than $\frac{1}{2^n}$ when $x = n - 1$. Then prove that the series converges for all x and satisfies the required inequalities. Either insure that the monomials are distinct or handle what could happen if they are not.