Math 503 **Problem Set 2** **9/18/2007**

Solutions are due at the beginning of class on Friday, September 28, 2007.

Chapter 2 of the text is probably most applicable to what we are doing this week, but there is also relevant information in chapter 11. We're proving a **very important theorem**, one of the major foundational results of the subject. Much or even most of the course material will follow from this result.

1. Please write a proof of Proposition 2.1.9*, that C^1 complex line integrals are independent of parameterization. This should be easy and brief. I can't verify the result in class because I get confused.

2. Problem 3 in chapter 2. This is the now-standard proof of the Poincaré Lemma applied to our situation.

3. Problem 4b in chapter 2. A calculation.

4. Problem 25 in chapter 2. Suggestion: don't try to "understand" these equations, just discover and verify some examples.

5. In this problem, g is a continuous complex-valued function defined on $[0,1]$, the closed unit interval. If $z \notin [0,1]$, define $f(z) = \int_0^1 \frac{g(t)}{t-z} dt$.

a) Prove that f is holomorphic in $\mathbb{C} \setminus [0,1]$ and obtain a formula for f'. Write the definition of f' as a limit and discover what the limit should be. You could imitate part of the text's proof of Theorem 3.1.1..

b) Prove that $\lim_{z \to \infty} f(z) = 0$**.

c) Suppose that $g(t) = 1$ on the unit interval. What is the resulting f? Does it satisfy the conclusion of b)? What about the behavior of f "across" $[0,1]$?

6. a) Suppose R is a closed rectangle with sides parallel to the coordinate axes in $\mathbb{C}$, and ∂R is its boundary oriented in the standard manner. Also suppose that f is a C^1 function defined in an open set containing R. Use the version of Green's Theorem quoted in class and a translation process (from x and y to z and $\overline{z}$) to verify that $\int_{\partial R} f(z)\, dz = 2i \int\int_R \frac{\partial f}{\partial \overline{z}}\, dxdy$ (p. 490 of the text).

b) (This is a simple version of Morera's Theorem, another criterion for holomorphicity.) Suppose that f is a C^1 function defined in an open subset, U, of $\mathbb{C}$. Suppose also that the line integral of f around the boundary of any rectangle contained in U is 0. Prove that f is holomorphic. If f is not holomorphic, then $\frac{\partial f}{\partial \overline{z}} \neq 0$ somewhere. With the complex Green's Theorem in mind, choose your rectangle so that …

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* There is a minor misprint in the text's statement.

** Given $\varepsilon > 0$, there is $M > 0$ so that if $|z| > M$ then $f(z)$ is defined and $|f(z)| < \varepsilon$.