Math 503 **Problem Set 3** **9/28/2007**

Solutions are due at the beginning of class on Tuesday, October 9, 2007.

Please finish reading chapter 2 of the text and begin reading chapter 3.

1. Suppose W is a subset of $\mathbb{C}$. If $s > 0$, define W_s to be $\bigcup_{w \in W} \overline{D_s(w)}$, where $D_s(w) = \{z \in \mathbb{C} \text{ with } |z - w| < s\}$. Suppose that K is a compact subset of an open set U of $\mathbb{C}$.

a) Prove that there is $t > 0$ so that if $0 < s \leq t$, then K_s is compact in U, and $(K_s)^o$ (the interior of K_s) contains K.

b) If s is such a number, and if $f \in \mathcal{O}(U)$, then $\|f'\|_K \leq C_s \|f\|_{K_s}$ where C_s is a positive constant which depends only on s and not on f or U or K. Here $\|g\|_W = \sup_{z \in W} |g(z)|$.

c) Is a similar inequality true for C^1 functions on $\mathbb{R}$? Verify or find counterexamples.

Thanks to S. Durst for essentially suggesting this problem.

2. a) Suppose f is holomorphic on an open subset of $\mathbb{C}$ which contains the unit interval $[0, 1]$ in $\mathbb{R}$. Give an example to show that the simplest version of the Mean Value Theorem is *not* true: find f and U with *no* $t \in [0, 1]$ so that $f'(t)(1 - 0) = f(1) - f(0)$.

b) Suppose f is holomorphic in a connected open subset of $\mathbb{C}$ containing $[0, 1]$. Is *some* version of MVT correct? So $f(1) - f(0) = f'$(a point in f's domain)(something reasonable) is the equation. For example, maybe this is true if the multiplier of the derivative is a path length*, and "somewhere" is a point on the path (path and point depending on f).

Thanks to D. Duncan for essentially suggesting this problem.

3. Recently some young scholars (!) wrote this: $\left|\sum_{n=1}^{\infty} f_n(s)\right| \leq \sum_{n=1}^{\infty} M_n$. In their context, the M_n's were a sequence of non-negative real numbers, and each f_n was a complex-valued function defined on some set, S. Also, the sum of the constants converged, and the sum of the functions converged for each $s \in S$.

a) Suppose that the inequality is correct. Show by example, however, that the scholars could *not* conclude that the sum of the functions converged uniformly on S.

b) Suppose the inequality is correct. Must the sum of the functions converge absolutely (pointwise)?

Thanks to several students for essentially suggesting this problem.

4. Show that $\frac{1}{1+|z|} - \frac{1}{2+|z|} + \frac{1}{3+|z|} - \frac{1}{4+|z|} + \cdots + \frac{(-1)^{n-1}}{n+|z|} + \cdots$ is not absolutely convergent but is uniformly convergent in the whole complex plane.

From *Classical Complex Analysis* by Liang-sin Hahn and Bernard Epstein.

5. Show that $\sum_{n=1}^{\infty} \frac{z}{(1+|z|)^n}$ converges (absolutely) pointwise but not locally uniformly on $\mathbb{C}$.

From *Classical Complex Analysis* by Liang-sin Hahn and Bernard Epstein.

6. Let f be holomorphic in $D(0, R)$, $R > 1$. Calculate $\int_{\partial D(0,1)} \left(2 \pm (\zeta + \zeta^{-1})\right) \frac{f(\zeta)}{\zeta}\, d\zeta$ two different ways and thereby deduce that $\pi^{-1} \int_0^{2\pi} f(e^{it}) \cos^2\left(\frac{1}{2}t\right) dt = f(0) + \frac{1}{2}f'(0)$ and $\pi^{-1} \int_0^{2\pi} f(e^{it}) \sin^2\left(\frac{1}{2}t\right) dt = f(0) - \frac{1}{2}f'(0)$.

From *The Theory of Complex Functions* by Reinhold Remmert.

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* of a path connecting 0 and 1 **(added 10/5/2007, 3:05 PM)**