Math 503 **Problem Set 5** **10/26/2007**

Solutions are due at the beginning of class on Tuesday, November 13, 2007.

1. One of the definitions of the **dilogarithm** Li_2 is the series $\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2}$.

a) Determine its radius of convergence, R. Does the series converge on $\overline{D(0,R)}$?
b) For each real $\varepsilon > 0$, determine an integer n_ε such that for every integer $m \geq n_\varepsilon$ and for every $z \in D(0,R)$, $\left| \sum_{n>m} \frac{z^n}{n^2} \right| < \varepsilon$.
c) Show that inside the topological interior of the disc of convergence the complex dilogarithm satisfies a second-order linear ordinary differential equation with rational coefficients.

From *Complex Analysis in One Variable* (second edition) by Raghavan Narasimhan and Yves Nievergelt.

2. a) Explain why there is an entire function which should be $\cos(\sqrt{z})$.
b) Suppose h is defined by $h(z) = \sqrt{\cos(z)}$. Find a disc of largest possible positive radius R so that h is holomorphic in $D_R(0)$. Explain why h cannot be extended to be holomorphic in a disc of larger radius centered at 0.

3. a) An entire function is of *exponential type* if there are $A > 0$ and $C > 0$ so that for all $z \in \mathbb{C}$ with $|z| > C$, $|f(z)| \leq Ce^{A|z|}$. Prove that the collection of entire functions of exponential type is closed under differentiation. In fact, if f is of exponential type, so is f', with the same A but with a possibly different C.
b) Show that an analogous statement for C^∞ functions on $\mathbb{R}$ is false. That is, define a C^∞ real-valued function g on $\mathbb{R}$ to be of *exponential type* if there are $A > 0$ and $C > 0$ so that for all $x \in \mathbb{R}$ with $|x| > C$, $|f(x)| \leq Ce^{A|x|}$. Give an example of a C^∞ function on $\mathbb{R}$ of exponential type whose derivative is **not** of exponential type for *any* choices of C and A.

4. (20 points, since this problem has many parts) Suppose U is open in $\mathbb{R}$. A function $f\colon U \to \mathbb{R}$ is *real analytic* or C^ω if, for every $a \in U$, there is $\delta > 0$ with $(a-\delta, a+\delta) \subseteq U$ and there is a sequence of real numbers $\{c_n\}$ so that $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all $x \in (a-\delta, a+\delta)$: f is real analytic if f is locally the sum of a real power series. (The coefficients c_n may depend upon both f and a.)
a) Here Ω is an open and connected subset of $\mathbb{C}$. Suppose $F\colon \Omega \to \mathbb{C}$ is holomorphic and $F(\Omega \cap \mathbb{R}) \subseteq \mathbb{R}$. If $U = \Omega \cap \mathbb{R}$ and $f = F\big|_U$, prove that $f\colon U \to \mathbb{R}$ is real analytic.
b) Suppose $f\colon U \to \mathbb{R}$ is real analytic with U open and connected in $\mathbb{R}$. Prove that there is an open and connected subset Ω of $\mathbb{C}$ and a complex analytic function $F\colon \Omega \to \mathbb{C}$ so that $U = \Omega \cap \mathbb{R}$ and $f = F\big|_U$. (F is called a *complexification* of f.)
c) Find an f which is C^∞ on $\mathbb{R}$ but *not* real analytic on $\mathbb{R}$.
d) Find an f which is real analytic on $\mathbb{R}$ whose Taylor series at 0 has finite radius of convergence.
e) Liouville's Theorem is false for real analytic functions: find an f which is real analytic on $\mathbb{R}$, non-constant, and bounded. Why does such an example exist?
f) An "entire" real analytic function may not have a complexification with domain $\mathbb{C}$. If $\varepsilon > 0$ define $S_\varepsilon \subset \mathbb{C}$ by $S_\varepsilon = \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \varepsilon\}$. Construct a real analytic function $f\colon \mathbb{R} \to \mathbb{R}$ so that there is *no* holomorphic F_ε defined on S_ε with $F_\varepsilon\big|_{\mathbb{R}} = f$ for *any* $\varepsilon > 0$.

OVER!!!

Midterm Exam Preparation

The Math 503 midterm exam will be given in class on **Tuesday, November 13**. The purpose of the exam is to assess your knowledge of complex variables and to prepare you for our written qualifying exam. Two or three of the following problems will appear on the exam, which will have five or six problems.

1. Suppose that σ is a permutation of the positive integers, $\mathbb{N}$ (so $\sigma : \mathbb{N} \to N$ is a bijection, that is, one-to-one and onto), and that $\sum_{j=1}^{\infty} a_j$ converges absolutely and its sum is S. Prove that $\sum_{j=1}^{\infty} a_{\sigma(j)}$ converges, and that the sum of this series is S.

2. Problem 41 of Chapter 2 of the textbook.

3. Suppose $f(z) = \frac{1}{z}$. What is $\inf_{P \in \mathbb{C}[z]} \{\sup_{z \in \partial D_1(0)} |f(z) - P(z)|\}$?

4. Problem 1 of Chapter 4 of the textbook.

5. Suppose W is a discrete closed subset of a connected open subset, U, of $\mathbb{C}$. Prove that W is countable (it may be finite or infinite, but it is countable!) and that $U \setminus W$ is connected.