Math 503 **Problem Set 6** **11/15/2007**

Solutions are due at the beginning of class on Tuesday, December 4, 2007.

1. Prove there is no proper holomorphic mapping from the unit disc to the complex plane.

2. If the Taylor series of an entire function converges to the function *uniformly*, then the function must be a polynomial. So uniform convergence on all of $\mathbb{C}$ is wrong, wrong, wrong ...

3. Use the Residue Theorem to compute $\displaystyle\int_0^{2\pi} \frac{1}{2+(\sin\theta)^2}\,d\theta$.

4. Suppose n is a positive integer. Use the Residue Theorem to compute $\displaystyle\int_0^{\infty} \frac{1}{1+x^{2n}}\,dx$♠. Check that the answer you get has the correct limiting behavior as $n \to \infty$.

5. Do either a) or b).

a) (5 points) Show that there is no function f holomorphic in the punctured disc $D^* = D_1(0)\setminus\{0\}$ satisfying $f(z)^2 = z$ for all $z \in D^*$. (So $\sqrt{z}$ doesn't *exist* in $\mathcal{O}(D^*)$.)

b) (10 points) Show that there is no function f holomorphic in the annulus A, the set of $z \in \mathbb{C}$ with $1 < |z| < 2$, satisfying $f(z)^2 = z$ for all $z \in A$. (So $\sqrt{z}$ doesn't *exist* in $\mathcal{O}(A)$.)

6. (After we do Rouché's Theorem!) I received e-mail about 6 months ago from a fairly recent Ph.D. graduate of our program who specialized in an aspect of combinatorics♡. Here is part of the message, as received:

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You see, I have a family of polynomials
x^(k+1) * (1+2x2-x3) + 2x - 4.
They all have a real root somewhere between 2 and 2.20556...    This
even I can prove.  But what I really need to prove (extra credit if
you can guess why) is that that is their root of greatest modulus.
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Please help this person, who even prepared some supporting evidence (pictures of approximate locations of the roots – see the course web page)♢.

7. (After we do Schwarz's Lemma!) Let $f : D(0,1) \to H$♣ be holomorphic with $f(0) = i$. If $f(0) = i$, prove that

a) $\frac{1-|z|}{1+|z|} \le |f(z)| \le \frac{1+|z|}{1-|z|}$ for $z \in D(0,1)$.

b) $|f'(0)| \le 2$.

From *Theory of Complex Functions* by Reinhold Remmert.

♠ The case n=1 was done in class. You can arrange this so only *one* residue must be computed (hint: Naqvi!).

♡ What else?

♢ Isn't this better than some abstract textbook example?

♣ $D(0,1)$ is the unit disc and H is the upper halfplane.