Math 507: Functional Analysis (Spring, 2004)

A1 Suppose L is a closed subspace of a Hilbert space, H. If $h \in H$, define $P(h)$ to be the element of L which is closest to h. Use results from class to verify the following:

a) Show that $P(h)$ is well-defined.

b) Show that $P\colon H \to H$ is linear and continuous, with $\|P\| =?$ and $P \circ P = P$.

c) Show that $\ker P = L^{\perp}$ is the collection of vectors orthogonal to L and that $\operatorname{im} P = L$.

Remarks Problem 3 defines $L^{\perp}$. P is called the *orthogonal projection* onto L. The facts stated in this problem are standard parts of Hilbert space technique. See if you can prove them without looking at a reference.

A2 Suppose $\sum a_j$ is a series of real numbers. This problem discusses the relationship between $\sum_{j\in\mathbb{N}} a_j$, the unordered sum defined in class, and the traditional $\sum_{j=1}^{\infty} a_j = \lim_{J\to\infty} \sum_{j=1}^{J} a_j$.

a) Show by (verified!) example that $\sum_{j=1}^{\infty} a_j$ may converge but that $\sum_{j\in\mathbb{N}} a_j$ may not.

b) Prove that $\sum_{j\in\mathbb{N}} a_j$ converges if and only if $\lim_{J\to\infty} \sum_{j=1}^{J} |a_j|$ exists.

A3 If $S \subseteq H$, a non-empty subset of a Hilbert space, $S^{\perp} = \{w \in H : \langle w, s\rangle = 0\,\forall s \in S\}$.

a) Show that $S^{\perp}$ is always a closed linear subspace of H. Can $S^{\perp}$ be $\{0\}$? Can it be H?

b) Show that $(S^{\perp})^{\perp}$ is the closure of the linear span of S.

A4 Let X be the vector space of all trigonometric polynomials on the real line: these are functions of the form $f(t) = \sum_{j=1}^{n} c_j e^{is_j t}$ where $s_j \in \mathbb{R}$ and $c_j \in \mathbb{C}$. Show that

$$\langle f, g\rangle = \lim_{A\to\infty} \frac{1}{2A} \int_{-A}^{A} f(t)\overline{g(t)}\, dt$$

is an inner product on X, that

$$\|f\|^2 = \langle f, f\rangle = \sum_{j=1}^{n} |c_j|^2$$

and that the completion of X is a nonseparable Hilbert space, H. Show that H contains all *uniform* limits of trigonometric polynomials; these are the so-called "almost-periodic" functions on $\mathbb{R}$.

Reference This is problem 29 of Chapter 12 of Rudin's *Functional Analysis.*

A5 Suppose H is a Hilbert space, and $T\colon H \to \mathbb{F}$ is a linear map. Prove that T is continuous if and only if $\ker T$ is a closed subspace of H. If T is *not* continuous, show that $\ker T$ is a dense subset of H.

A6 Either compute the Bergman kernel explicitly for some *other* domain in $\mathbb{C}$ (half plane, strip, annulus) or, if possible, compute some sort of Bergman-like kernel for, say, L^2 harmonic functions on the unit disc and verify it has similar properties.