Math 507: Functional Analysis (Spring, 2004)

B1 Suppose K is a compact non-empty subset of $\mathbb{C}$. Show that there is $T \in L(H)$ so that $\sigma(T) = K$.

B2 The *Volterra operator*, $V\colon L^2([0,1]) \to L^2([0,1])$, is defined by $Vf(x) = \displaystyle\int_0^x f(t)\,dt$.

a) What is V^*? $V + V^*$? What is the image of $V + V^*$?

Remark This is problem 7 of section 2.2 in Conway's *A Course in Functional Analysis*.

b) What is $\sigma_p(V)$, the collection of eigenvalues of V?

B3 (Continuing the preceding problem.) What can you say about $\sigma(V)$? (You will probably need the *Spectral Radius Formula* and inductive discussion of V^n and $\|V^n\|$.)

B4 (Hilbert matrix) Show that $\langle Ae_j, e_i\rangle = (i+j+1)^{-1}$ for $0 \le i, j < \infty$ defines a bounded operator on $\ell^2(\mathbb{N} \cup \{0\})$ (square-summable sequences beginning with the index 0) with $\|A\| \le \pi$.

Remark This is problem 10 of section 2.1 in Conway's *A Course in Functional Analysis*. The problem statement there contains further references.

B5 If H is an infinite dimensional Hilbert space, show that no orthonormal basis for H is a Hamel (vector space) basis. Show that a Hamel basis is uncountable.

B6 Suppose H is the collection of all absolutely continuous functions $f(0)\colon [0,1] \to \mathbb{F}$ with $f(0) = 0$ and $f' \in L^2([0,1])$. Let $\langle f, g\rangle = \int_0^1 f'(t)\overline{g'(t)}\,dt$.

a) Prove that H is a Hilbert space.

b) Find an orthonormal basis of H.

Remark This is problem 3 of section 1.1 and problem 4 of section 1.4 in Conway's *A Course in Functional Analysis*.

B7 Let $H = \ell^2(\mathbb{N} \cup \{0\})$ (square-summable sequences beginning with the index 0).

a) Show that if $\{\alpha_n\} \in H$, then the power series $\sum_{n=0}^{\infty} \alpha_n z^n$ has radius of convergence ≥ 1.

b) If $|\lambda| < 1$ and $L\colon H \to \mathbb{F}$ is defined by $L(\{\alpha_n\}) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$, find the vector h_0 in H so that $L(h) = \langle h, h_0\rangle$ for all $h \in H$. What is the norm of L?

c) Define $\tilde{L}\colon H \to \mathbb{F}$ by $\tilde{L}(\{\alpha_n\}) = \sum_{n=1}^{\infty} n\alpha_n \lambda^{n-1}$, again with $|\lambda| < 1$. Now find the corresponding $\tilde{h_0}$ so that $\tilde{L}(h) = \langle h, \tilde{h_0}\rangle$ for all $h \in H$.

Remark This is problem 3 and problem 4 of section 1.3 in Conway's *A Course in Functional Analysis*.

B8 Suppose that A and B are self-adjoint. Prove that AB is self-adjoint if and only $AB = BA$.

Remark This is problem 11 of section 2.3 in Conway's *A Course in Functional Analysis*.