February 12, 2004

Math 507: Functional Analysis (Spring, 2004)

C1 Suppose $T \in L(H)$. Prove that if T is compact then T^* is compact.

C2 Let $\mathcal{C}$ be the compact operators in $L(H)$. Suppose $\mathcal{I}$ is a two-sided closed ideal of $L(H)$. If $\mathcal{I} \neq \{0\}$ then $\mathcal{C} \subseteq \mathcal{I}$.

C3 Prove that *connectedness* is not axiomatizable in the language of graphs.

C4 a) Suppose that $\{\phi_n\}_{n\in\mathbb{N}}$ is a complete orthonormal sequence in $L^2([0,1])$. Is the set $\{\phi_n(x)\overline{\phi_m(y)}\}_{(n,m)\in\mathbb{N}^2}$ a complete orthonormal set in $L^2([0,1]^2)$?

b) Is a similar result true for all L^2 spaces?

C5 a) Define the right-shift, S on $\ell^2(\mathbb{N})$ by $S(e_j) = e_{j+1}$. Is S compact? What is $\sigma_p(S)$? What is $\sigma(S)$?

b) Do the same for S^*.

C6 a) Suppose $\{T_n\}$ is a sequence in $L(H)$ which converges to $T \in L(H)$. If $z_n \in \sigma(T_n)$ and $z_n \to z \in \mathbb{C}$, prove that $z \in \sigma(T)$.

b) Find a simple example where $T_n \to T$ but the sequence $\{z_n\}$ does *not* converge.

Remark This is essentially problem 5 of section 7.3 in Conway's *A Course in Functional Analysis*.

C7 Suppose S and T are in $L(H)$ and $\lambda \in \rho(ST)$ with $\lambda \neq 0$. Prove that $\lambda \in \rho(TS)$ and $(\lambda I - TS)^{-1} = \lambda^{-1}I + \lambda^{-1}T(\lambda I - ST)^{-1}S$. Show that $\sigma(ST) \cup \{0\} = \sigma(TS) \cup \{0\}$ and give an example such that $\sigma(ST) \neq \sigma(TS)$.

Remark This is essentially problem 7 of section 7.3 in Conway's *A Course in Functional Analysis*.

C8 Let P and Q be orthogonal projections onto subspaces M and N of a Hilbert space H. Suppose that $PQ = QP$.

a) Prove that $I-P$, $I-Q$, PQ, $P+Q-PQ$, and $P+Q-2PQ$ are orthogonal projections.

b) How are the ranges of the projections in a) related to M and N?

C9 Let P and Q be orthogonal projections onto subspaces M and N in a Hilbert space H. Prove that $\lim_{n\to\infty} (PQ)^n$ exists and is the orthogonal projection onto $M \cap N$.

Remark The last two problems are from chapter 6 of Reed and Simon's *Methods of Modern Mathematical Physics, I: Functional Analysis*. The second problem is "starred", indicating it is a "harder" problem, "included ... in order to challenge the reader."