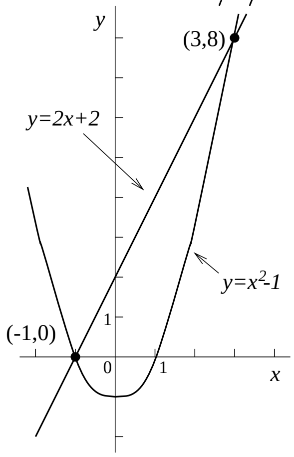
640:135:16-18 **Answers to version A of Exam #1** **10/9/97**

(16) 1. The parabola $y = x^2 - 1$ and the line $y = 2x + 2$ intersect in two points.

- Use algebra to find the coordinates of these points.
- Sketch both curves on the indicated axes.
- Label the curves and label the coordinates of the points where they intersect.



Answer Since the y coordinates must be equal, we get $x^2 - 1 = 2x + 2$ which gives $x^2 - 2x - 3 = 0$ which gives $(x-3)(x+1) = 0$, so roots are $+3$ and -1. When $x = 3$, $y = 3^2 - 1 = 9 - 1 = 8$ or $y = 2 \cdot 3 + 2 = 6 + 2 = 8$ (it is cheap to check that no mistake has been made by substituting in both equations). When $x = -1$, $y = (-1)^2 - 1 = 1 - 1 = 0$ or $y = 2 \cdot (-1) + 2 = -2 + 2 = 0$. The points of intersection are $(3, 8)$ and $(-1, 0)$.

(16) 2. Suppose that the function F is described by

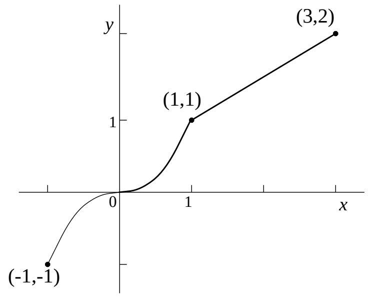
$$F(x) = \begin{cases} x^3, & \text{if } -1 \le x \le 1 \\ \frac{1}{2}x + A, & \text{if } 1 < x \le 3 \end{cases}.$$

a) Find A so that F is continuous for all numbers in its domain. Explain your answer.

Answer $x = 1$ is the candidate for discontinuity. The left- and right-hand limits of F at 1 must agree with F's value at 1. $F(1) = 1$. $\lim\limits_{x\to 1^-} F(x) = \lim\limits_{x \to 1^-} x^3 = 1^3 = 1$. $\lim\limits_{x \to 1^+} F(x) = \lim\limits_{x\to 1^+} \frac{1}{2}x + A = \frac{1}{2} + A$. So we have $\frac{1}{2} + A = 1$ or $A = \frac{1}{2}$.

b) Graph $y = F(x)$ for the value of A found in a).

Comment Dots have been added for emphasis at the "edges" of each formula's interval of definition. The points have been labeled which is helpful in part c).



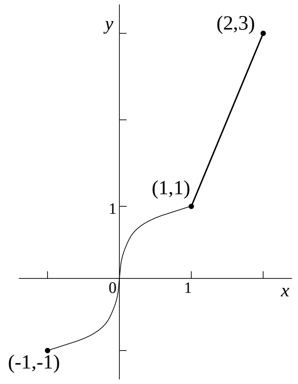
c) Suppose G is the function inverse to F with the value of A found in a). Describe G using one or more formulas: be as precise as you can with G's algebraic description and with its domain.

Answer The inverse to x^3 is certainly $x^{1/3}$. For the formula $y = \frac{1}{2}x + \frac{1}{2}$ we multiply by 2 and get $2y = x + 1$ and then $2y - 1 = x$, which gives a formula for G in the "other" interval. So:

$$G(x) = \begin{cases} x^{1/3}, & \text{if } -1 \le x \le 1 \\ 2x - 1, & \text{if } 1 < x \le 2 \end{cases}.$$

Note that it doesn't matter which formula "owns" the domain value 1, since $G(1)$ will be the same.

d) Graph $y = G(x)$.



(20) 3. Evaluate the indicated limits exactly. Give evidence to support your answers.

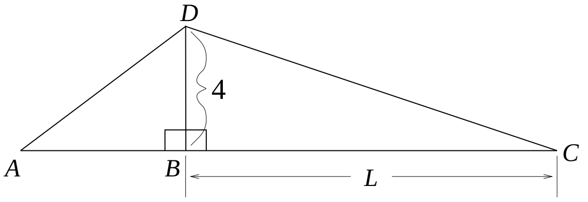
a) $\lim\limits_{x\to 5} \frac{x-5}{x^2-25}$ **Answer** If $x \ne 5$ we know $\frac{x-5}{x^2-25} = \frac{x-5}{(x+5)(x-5)} = \frac{1}{x+5}$. But $\frac{1}{x+5}$ is continuous at 5, so we evaluate $\lim\limits_{x\to 5} \frac{1}{x+5}$ by "plugging in" (substituting) and get $\frac{1}{10}$.

b) $\lim\limits_{x \to 4} \frac{\sqrt{x}-2}{x-4}$ **Answer** If $x \ne 4$ we know $\frac{\sqrt{x}-2}{x-4} = \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \frac{x-4}{(x-4)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2}$. But $\frac{1}{\sqrt{x}+2}$ is continuous at 4, so we evaluate $\lim\limits_{x\to 4} \frac{1}{\sqrt{x}+2}$ by substituting, and get $\frac{1}{4}$.

c) $\lim\limits_{x \to 3^-} \frac{x^2+1}{x+2}$ **Answer** $\frac{x^2+1}{x+2}$ is already continuous at 3. Therefore standard limit theorems can be used ("plugging in"). The limit is $\frac{3^2+1}{3+2}$, a fine answer. Or: $\frac{10}{5} = 2$.

d) $\lim\limits_{x \to 2^-} \frac{x-3}{x-2}$ **Answer** Here, as $x \to 2^-$, x is close to 2, and $x < 2$. Then $x - 3 \approx -1$, and $x - 2$ is small and negative. So $\frac{x-3}{x-2} \approx \frac{-1}{\text{small, negative}} = \text{large, positive}$. Therefore $\lim\limits_{x\to 2^-} \frac{x-3}{x-2} = +\infty$. Graphical verification of this is also possible.

(10) 4. The following facts should be assumed in all parts of this problem:
The line ABC in the picture is a straight line, and the segment BD is perpendicular to the line, so triangles $\triangle ABD$ and $\triangle BCD$ are right triangles. The length of the line segment BD is 4, and the length of the line segment BC is L.



• Write a formula in terms of an inverse trigonometric function for $\angle DCB$ in radians. Your answer should use 4 and L. **Answer** $\tan(\angle DCB) = \frac{4}{L}$.
Therefore **ANSWER:** $\arctan\left(\frac{4}{L}\right)$

• Now suppose that $\angle BAD$ is equal to twice $\angle DCB$. What is $\angle BAD$ in radians? Use the answer you have above. **ANSWER:** $2\arctan\left(\frac{4}{L}\right)$

• What is the length of the line segment AD? Use the answer you have above and a trigonometric function.

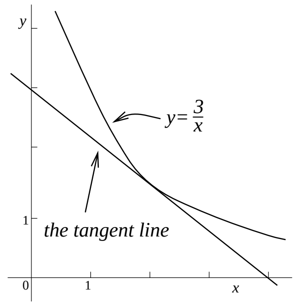
Answer $\sin(\angle BAD) = \frac{4}{AD}$ so $AD = \frac{4}{\sin(\angle BAD)}$ **ANSWER:** $\frac{4}{\sin(2\arctan\left(\frac{4}{L}\right))}$

Comment No "simplification" is necessary. False simplifications might be tempting, but neither sin nor arctan are linear functions – the answer is a composition of non-linear functions.

(16) 5. a) Write the definition of derivative as a limit and *use this definition* to find the derivative of $f(x) = \frac{3}{x}$.
Answer $f'(x) = \lim_{\triangle x \to 0} \frac{f(x+\triangle x)-f(x)}{\triangle x}$. We simplify this with our specific f when $\triangle x \neq 0$:
$\frac{f(x+\triangle x)-f(x)}{\triangle x} = \frac{\frac{3}{x+\triangle x}-\frac{3}{x}}{\triangle x} = \frac{3}{\triangle x}\left(\frac{1}{x+\triangle x}-\frac{1}{x}\right) = \frac{3}{\triangle x}\left(\frac{x-(x+\triangle x)}{(x+\triangle x)x}\right) = \frac{3}{\triangle x}\left(\frac{-\triangle x}{(x+\triangle x)x}\right) = \frac{-3}{(x+\triangle x)x}$. The last algebraic form allows us to find the limit: $f'(x) = \lim_{\triangle x \to 0} \frac{-3}{(x+\triangle x)x} = \frac{-3}{x^2}$.

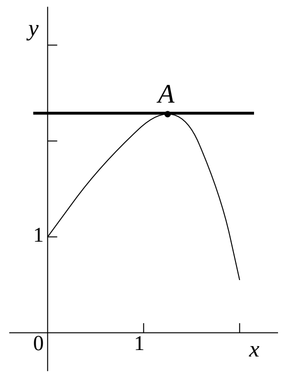
b) Use your answer to a) to find an equation for the line tangent to the curve $y = \frac{3}{x}$ when $x = 2$.
Answer The slope of the tangent line is $f'(2) = -\frac{3}{4}$ and the line goes through the point $(2, f(2))$ which is $(2, \frac{3}{2})$. Therefore an equation for the line is $y - \frac{3}{2} = -\frac{3}{4}(x-2)$. This is a fine answer. If you wish you can convert it to slope-intercept form: $y = -\frac{3}{4}x + 3$.

c) Sketch the graph of $y = \frac{3}{x}$ *and* the line you found in answer to b) on these coordinate axes: Label the curves you draw.



(12) 6. a) If $f(x) = 3x^5 - 8x^2 + 4$, what is $f'(x)$? **Answer** $f'(x) = 15x^4 - 16x$.
b) If $g(x) = \frac{3x-2}{2x-3}$, what is $g'(x)$? **Answer** $g'(x) = \frac{3(2x-3)-2(3x-2)}{(2x-3)^2}$. Another correct answer is $\frac{-13}{(2x-3)^2}$.
c) If $h(x) = x^2 \sin x$, what is $h'(x)$? **Answer** $h'(x) = 2x\sin x + x^2\cos x$.
d) If h is the function in part c), compute $h(\pi)$ and $h'(\pi)$ exactly, using exact values of any trigonometric functions involved. **Answer** $h(\pi) = \pi^2 \sin\pi = 0$ and $h'(\pi) = 2\pi\sin\pi + \pi^2\cos\pi = -\pi^2$.

(10) 7. The graph of the function $f(x) = 3x - e^x + 2$ in the window $0 \le x \le 2$ and $0 \le y \le 3$ is shown below:
The point A is the highest point on this graph.



a) Draw the tangent line to the graph at the point A on the graph given above. What is the slope of the line you have drawn? Your answer should be very simple. **ANSWER:** 0

b) Use calculus to find the two coordinates of the point A. The coordinates of A should be given exactly, using expressions like "$3\sqrt{5}-6\ln 2$." Computations with exp and log should be simplified as much as possible. A numerical approximation like 1.40135 is **not** acceptable.
Answer The slope of the tangent line is 0 where $f'(x) = 0$. But $f'(x) = 3 - e^x$. This is 0 when $3 - e^x = 0$ or $3 = e^x$ or $\ln 3 = x$. $f(\ln 3) = 3\ln 3 - e^{\ln 3} + 2 = 3\ln 3 - 3 + 2 = 3\ln 3 - 1$. The coordinates of A are $(\ln 3, 3\ln 3 - 1)$.