640:135:16-18 **Answers to version B of Exam #2** **11/18/97**

(17) 1. Find $\frac{dy}{dx}$: **Note** I'll use y' for $\frac{dy}{dx}$ in this problem.

a) $y = 3e^{2x} - 5x\ln(x^2+1)$ **Answer** $y' = 3e^{2x}\cdot 2 - 5\ln(x^2+1) - 5x\cdot\frac{2x}{x^2+1}$.

b) $y = \sin\left(\frac{x^2+3}{x^2-1}\right)$ **Answer** $y' = \cos\left(\frac{x^2+3}{x^2-1}\right)\cdot\frac{(2x)\cdot(x^2-1)-(2x)\cdot(x^2+3)}{(x^2-1)^2}$.

c) $y^2 - x^3y + 2xy^5 = 4$ **Answer** $\frac{d}{dx}$ the equation to get: $2yy' - 3x^2y - x^3y' + 2y^5 + 2x(5y^4y') = 0$. We solve for y' : first write $(2y - x^3 + 10xy^4)y' - 3x^2y + 2y^5 = 0$ and then $y' = \frac{-(-3x^2y+2y^5)}{(2y-x^3+10xy^4)}$.

(17) 2. In this problem the function G is defined by the formula

$$G(x) = (2 - e^x)(1 - e^x)$$

a) What is $\lim_{x\to-\infty} G(x)$? What is $\lim_{x\to+\infty} G(x)$? **Answer** We know $\lim_{x\to-\infty} e^x = 0$ and $\lim_{x\to+\infty} e^x = \infty$. Use this to compute G's limits: $\lim_{x\to-\infty} G(x) = (2-0)\cdot(1-0) = 2$ and $\lim_{x\to+\infty} G(x) = (2-\text{large positive})\cdot(1-\text{large positive}) = +\infty$.

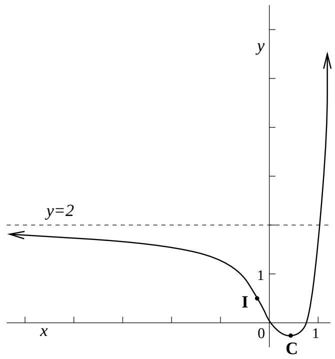
b) For which x is $G(x) = 0$? Computations with exp and log should be simplified as much as possible. A numerical approximation like 1.40135 is **not** acceptable. **Answer** Roots occur when either factor is 0: $2 - e^x = 0$ when $x = \ln 2$ and $1 - e^x = 0$ when $x = 0$. So the answer is: $\ln 2$ and 0.

c) Compute $G'(x)$. For which x is $G'(x) = 0$? Computations with exp and log should be simplified as much as possible. A numerical approximation like 1.40135 is **not** acceptable. **Answer** $G'(x) = (-e^x)(1-e^x) + (2-e^x)(-e^x) = -e^x(1-e^x+2-e^x) = -e^x(3-2e^x)$. Since the $-e^x$ factor can never be 0, G' will vanish whenever $3 - 2e^x = 0$ which is when $\frac{3}{2} = e^x$ or when $x = \ln\frac{3}{2}$.

d) Compute $G''(x)$. For which x is $G''(x) = 0$? Computations with exp and log should be simplified as much as possible. A numerical approximation like 1.40135 is **not** acceptable. **Answer** $G''(x) = (-e^x)(1-e^x) + (-e^x)(-e^x) + (-e^x)(-e^x) + (2-e^x)(-e^x)$ by using the product rule twice. Factor out one $-e^x$ term since that can never be 0. So $G''(x) = 0$ when $(1-e^x) + (-e^x) + (-e^x) + (2-e^x) = 0$ which happens if $3 = 4e^x$ or $x = \ln\frac{3}{4}$.

e) Sketch a graph of $y = G(x)$, using the information you have obtained in the previous parts of this problem. Indicate precisely any relative maxima or minima or points of inflection with exact coordinates, and any vertical or horizontal asymptotes with precise equations. Computations with exp and log should be simplified as much as possible. A numerical approximation like 1.40135 is **not** acceptable.

Answer The only critical number is $\ln\frac{3}{2}$ which has corresponding G-value equal to $(2-e^{\ln\frac{3}{2}})(1-e^{\ln\frac{3}{2}}) = (2-\frac{3}{2})(1-\frac{3}{2}) = -\frac{1}{4}$. The point corresponding to these numbers is labelled **C** on the graph and is a relative minimum. If $x = \ln\frac{3}{4}$ then G's value is $(2-e^{\ln\frac{3}{4}})(1-e^{\ln\frac{3}{4}}) = (2-\frac{3}{4})(1-\frac{3}{4}) = \frac{5}{16}$. The point corresponding to this pair of numbers is labelled **I** on the graph and is an inflection point. The equation $y = 2$ indicates the horizontal asymptote (only on the left!) of the graph.



(16) 3. A rectangular box with *no* top has height H and a square base with sides each S in length. Its volume is $\frac{4}{27}$ cubic feet. Find the specific values of S and H so that the box will have the *least* total surface area. Explain briefly why you have found the box with the least surface area.

Answer The volume of this box is S^2H and the variables in this problem are constrained by $S^2H = \frac{4}{27}$. The surface area, A, is the sum of the area of the bottom (S^2), plus the area of the four sides, each of which is a rectangle of area SH. So $A = S^2 + 4SH$ is the objective function to be minimized. The constraint can be used to reduce A's formula to 1 variable, since $H = \frac{4}{27S^2}$. Therefore $A = S^2 + 4S\cdot\frac{4}{27S^2} = S^2 + \frac{16}{27S}$. Let's look for critical numbers: $A' = 2S - \frac{16}{27S^2}$. This is 0 when $2S = \frac{16}{27S^2}$ which happens when $S^3 = \frac{8}{27}$ so $S = \frac{2}{3}$ is the only critical number. The corresponding H is gotten from the constraint equation, so $H = \frac{4}{27(2/3)^2} = \frac{1}{3}$. Why do these values represent a minimum? Here are some possible reasons (just one needs to be given!):

The function level $\lim_{S\to0^+} A = +\infty$ and $\lim_{S\to+\infty} A = +\infty$ follow easily from the formula for A. So $S = \frac{2}{3}$ must represent a minimum.
THE ZEROTH(?) DERIVATIVE TEST

The first derivative level Since $A' = 2S - \frac{16}{27S^2}$, when S is to the left of $\frac{2}{3}$ (near 0) $A' < 0$ and when S is very large positive, $A' > 0$. Therefore A decreases on the left of $\frac{2}{3}$ and increases on the right of $\frac{2}{3}$. So $\frac{2}{3}$ must represent a minimum.
THE FIRST DERIVATIVE TEST

The second derivative level $A'' = 2 + \frac{32}{27S^3}$ and this is positive when $S = \frac{2}{3}$. Thus A is concave up, so $\frac{2}{3}$ represents a minimum.
THE SECOND DERIVATIVE TEST

(22) 4. Suppose that $F'(x)$, **the derivative** of F, is given by the formula

$$F'(x) = x^{30}(x-1)^{57}$$

Do **not** try to find a formula for F itself! That is difficult to do and will not help you complete this problem.

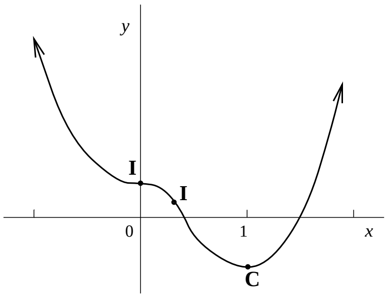
a) For which x is $F'(x) = 0$? For which x is $F'(x) > 0$? For which x is $F'(x) < 0$?
Answer F' is 0 at 0 and 1. Since 30 is even and 57 is odd, the sign of F' does not change at 0 and does change at 1. So $F' > 0$ when $x > 1$ and $F' < 0$ when $x < 0$ and $0 < x < 1$.

b) What are the critical numbers of F? Which of these are relative maxima and which are relative minima and which are neither? Give evidence to support your assertions.
Answer 0 and 1 are the critical numbers of F. 1 represents a relative minimum because the sign of F' changes at 1, from negative to positive (so F decreases to the left of $x = 1$ and then increases to the right). 0 is neither a relative max nor a relative min, since there is *no* sign change in F' at 0.

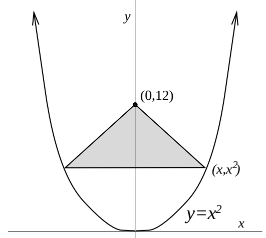
c) Find a formula for $F''(x)$. For which x is $F''(x) = 0$? For which x is $F''(x) > 0$? For which x is $F''(x) < 0$?
Answer $F''(x) = 30x^{29}(x-1)^{57} + x^{30}57(x-1)^{56} = x^{29}(x-1)^{56}\big(30(x-1) + 57x\big)$. Here the algebra is almost essential to recognize the simplicity of F''. In particular, it allows the roots of $F''(x) = 0$ to be seen: $x = 0$ and $x = 1$ and $x = \frac{30}{87}$. The last root comes from solving $30(x-1) + 57x = 0$. There are sign changes in F'' at 0 and $\frac{30}{87}$ but not at 1. Putting it all together: $F'' > 0$ when $x > 1$ and $\frac{30}{87} < x < 1$ and when $x < 0$. $F''0$ when $0 < x < \frac{30}{87}$.

d) What are the points of inflection of F? Why are they points of inflection? Give evidence to support your assertions.
Answer F has points of inflection at $x = 0$ and $x = \frac{30}{87}$. At those numbers, F'' changes sign, so there is a change in concavity. $x = 1$ is *not* a point of inflection since F'' doesn't change sign there.

e) Use the information you have obtained in earlier parts of this problem to sketch a possible graph of $y = F(x)$. Make sure that your graph is qualitatively correct and uses all the information you have deduced in previous parts of this problem. Label any points of inflection, relative maxima, and relative minima.
Answer The two points labelled **I** are points of inflection, at $x = 0$ and $x = \frac{30}{87}$. There is a relative minimum at **C** when $x = 1$. The function is decreasing in the interval $(-\infty, 1)$ and increasing in $(1, \infty)$. It is concave up in $(-\infty, 0)$ and $(\frac{30}{87}, \infty)$ and concave down in $(0, \frac{30}{87})$.



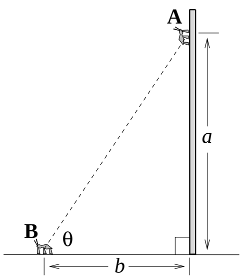
(15) 5. A triangle has one vertex at the point $(0, 12)$ and its opposite side, below it, is parallel to the $x-$axis with endpoints on the parabola $y = x^2$. Find all of the vertices of such a triangle which has the *largest* area. Explain briefly why you have found the largest such triangle.
Answer The area of a triangle is half the base times the altitude. Here the base is $2x$ and the altitude is $12 - x^2$. The area, A, is therefore $x(12 - x^2)$. In this case, $0 \leq x \leq \sqrt{12}$ and A is clearly 0 at the endpoints (the triangles there have 0 area). Since the area is positive between 0 and $\sqrt{12}$, a max must occur at a critical number. But $A' = 12 - 3x^2$ is 0 at ± 2, and only $x = 2$ is relevant. The vertices of the triangle of largest area are $(0, 12)$, $(2, 4)$, and $(-2, 4)$.



(13) 6. Ant **A** is crawling up a vertical pole at 0.3 feet per minute. At the same time ant **B** is crawling away from the base of the pole on the horizontal ground at 0.4 feet per minute.

a) If θ is the angle that ant **B** sees between the base of the pole and ant **A**, if a is the distance from ant **A** to the base of the pole, and if b is the distance from ant **B** to the base of the pole, write an equation connecting θ with a and b. You will need to use a trigonometric function. **Answer** $\tan\theta = \frac{a}{b}$.

b) Use the information provided to compute θ and $\frac{d\theta}{dt}$ at the instant that ant **A** is 10 feet up the pole and ant **B** is 5 feet from the base of the pole. Is the angle θ increasing or decreasing at this time?



Answer First, $\tan\theta = \frac{10}{5} = 2$, so $\theta = \arctan 2 \approx 1.107$ (the first answer is fine!). Now $\frac{d}{dt}$ the equation above, remembering to use the chain rule and the quotient rule: $(\sec\theta)^2\,\theta' = \frac{a'b - b'a}{b^2}$ (here $'$ means derivative with respect to time). Either remember that $(\sec\theta)^2 = 1 + (\tan\theta)^2 = 1 + 2^2 = 5$ or use your calculator to see that $(\sec 1.107)^2 \approx 4.997$. So the left-hand side "at the instant" is $5\theta'$. The right-hand side becomes $\frac{a'b - b'a}{b^2} = \frac{.3\cdot 5 - .4\cdot 10}{5^2} = -.1$ so we can solve for θ' and get $\theta' = \frac{-.1}{5} = -.02$ "at the instant". The angle θ is *decreasing* at this time.