(12) 1. Use Taylor's Theorem to explain how to compute $\cos(-2)$ to accuracy of $\pm.005$; you should write down a sum which is the desired approximation, and be sure to specify how you know that the error is within the desired tolerance.

Answer Taylor's Theorem states that $f(x) = T_n(x) + R_n(x)$ where $T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j$ and $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$ for some z between 0 and x. Here f is cos and all the derivatives of cos are either $\pm \sin$ or $\pm \cos$ and the absolute values of any of these functions can be overestimated by 1. Therefore the approximation (which is $T_n(-2)$) will be within $\pm.005$ if $|R_n(2)| < .005$ and $R_n(2)$ can be conveniently overestimated by $\frac{2^{(n+1)}}{(n+1)!}$. What n will make this less than .005? In another window on this screen, `maple` tells me that $\frac{2^9}{9!} \approx .0014$ so $n = 8$ works. The sum we want is $1 - \frac{(-2)^2}{2!} + \frac{(-2)^4}{4!} - \frac{(-2)^6}{6!} + \frac{(-2)^8}{8!}$.

(12) 2. What polynomial will approximate e^x with an error of $\pm.01$ on the interval $\left[-\frac{1}{5}, \frac{1}{5}\right]$? Be sure to explain how you know that your answer is correct.

Answer Taylor's Theorem states that $f(x) = T_n(x) + R_n(x)$ where $T_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j$ and $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$ for some z between 0 and x. Here f is the exponential function and all derivatives of exp are exp. The desired polynomial will be $T_n(x)$ and we need to discover an n so that $|R_n(x)| < .01$ for all $x's$ with $|x| \le \frac{1}{5}$. Since exp is an increasing function, e^z will be less than $e^{1/5}$. I'll overestimate this with 2 (surely the fifth power of 2 is bigger than e!). So $|R_n(x)| < \frac{2|x|^{n+1}}{(n+1)!}$. We want to overestimate this for all x's in the interval $\left[-\frac{1}{5}, \frac{1}{5}\right]$ so we can use $x = \frac{1}{5}$ in the estimate. Then we'll be successful if we can get $\frac{2\left(\frac{1}{5}\right)^{n+1}}{(n+1)!} < .01$. Again, `maple` reports that $\frac{2\left(\frac{1}{5}\right)^3}{3!} \approx .0026$ so $n = 2$ works. The polynomial we want is $1 + x - \frac{1}{2}x^2$.

(10) 3. What is $\lim_{x \to 0} \dfrac{\arctan(x^2) - (\arctan x)^2}{x^4}$? Don't use l'Hôpital's rule four times! Please don't.

Answer Let's use just an initial piece of the Taylor series for arctan (we probably don't need terms whose degrees are > 4). So $\arctan x = x - \frac{x^3}{3} +$ (Higher order terms than 4). Then $\arctan(x^2) = x^2 +$ (Higher order terms than 4). And $(\arctan x)^2 = (x - \frac{x^3}{3} + \text{(Higher order terms than 4)})^2 = x^2 + 2\frac{x^4}{3} +$ (Higher order terms than 4), so that $\frac{\arctan(x^2) - (\arctan x)^2}{x^4} = \frac{2}{3} +$ (Terms with at least one x in them). The limit we want (as $x \to 0$) is $\frac{2}{3}$. `maple`'s limit command reports the same answer.

(10) 4. Write $\int_0^2 e^{-x^2}\, dx$ as an infinite series using the Taylor series for the exponential function. Your answer should be in summation notation; indicate the general term as clearly as you can.

Answer Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we know that $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$. Then $\int_0^2 e^{-x^2}\, dx = \int_0^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}\Big]_0^2 = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)n!}$.

(10) 5. For which positive integers k does the series $\sum_{n=1}^{\infty} \dfrac{(n!)^k}{(kn)!}$ converge?

Comment: you may begin by considering a few "low" special cases like $k = 1$ and $k = 2$, but try as well as you can to give your reasoning in general.

Answer Let's try the ratio test. $a_n = \frac{(n!)^k}{(kn)!}$ so $a_{n+1} = \frac{((n+1)!)^k}{(k(n+1))!}$ and the quotient is $\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^k (kn)!}{(k(n+1))!(n!)^k} = \frac{(n+1)^k (kn)!}{(kn+k)!} = \frac{(n+1)^k}{(kn+k)(kn+k-1)\cdots(kn+1)}$. There are k terms in the bottom of the last fraction. Recognize that the top of the last fraction is a polynomial in n of degree k whose highest degree coefficient is 1 and the bottom is a polynomial in n of degree k whose highest degree coefficient is k^k. So the limit of $\frac{a_{n+1}}{a_n}$ as $n \to \infty$ is $\frac{1}{k^k}$. When $k > 1$ this is less than 1. Thus the series converges when $k > 1$. If $k = 1$ the ratio test provides no information, but the series itself is just $\sum_{n=1}^{\infty} 1$ then, so it diverges.

(10) 6. Show that $\sum_{n=2}^{\infty} \dfrac{1}{n \ln n}$ diverges. Also find a specific partial sum $S_N = \sum_{n=2}^{N} \dfrac{1}{n \ln n}$ with $S_N > 1{,}000$. Be sure to explain how you know that your answer is correct.

Answer The integral test says that this series will converge if $\int_2^\infty \frac{1}{x\ln x}\,dx$ is finite. We can use the substitution $u=\ln x$ (so $du=\frac{1}{x}\,dx$) and compute: $\int_2^A \frac{1}{x\ln x}\,dx = \ln(\ln(x))\big]_2^A = \ln(\ln(A)) - \ln(\ln(2)) \to \infty$ as $A\to\infty$ so the series diverges. A picture should convince you that S_N is greater than $\int_2^{N+1} \frac{1}{x\ln x}\,dx$. That integral is $\ln(\ln(x))\big]_2^{N+1} = \ln(\ln(N+1)) - \ln(\ln(2))$. `maple` remarked that direct manipulation of numbers to get this bigger than 1,000 is too hard (it declared: `Error, (in evalf/power) argument too large`) so I've got to think a bit. $\ln(\ln(2)) \approx -.366$. Any integer larger than $e^{\left(e^{1,000}\right)}$ (the exponentials are repeated since the ln's are repeated) will serve. Why not take $N = 10^{\left(10^{1,000}\right)}$?

(10) 7. Show that $\sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln n}$ converges. Also find a specific partial sum $S_N = \sum_{n=2}^{N} \frac{(-1)^n}{n\ln n}$ so that S_N is within $\frac{1}{1{,}000}$ of the sum of the whole series. Be sure to explain how you know that your answer is correct.

Answer The alternating series test applies (all of the hypotheses are satisfied). So the series converges. Since it *is* an alternating series, we know that the error committed by omitting the "infinite tail" is less than the first omitted term. When is $\frac{1}{n\ln n} < 1{,}000$? $n=200$ seems to work because $\frac{1}{200\ln(200)} \approx .00094$ so S_{200} will certainly be a close enough partial sum. You are *not* asked for the lowest suitable partial sum!

(14) 8. Consider the power series $\sum_{n=1}^{\infty}(n^2+3n+7)x^n$.

a) What is the radius of convergence of this series?

Answer The ratio test is easy to use. If $a_n = (n^2+3n+7)x^n$ then $a_{n+1} = ((n+1)^2+3(n+1)+7)x^{n+1}$ so that $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{((n+1)^2+3(n+1)+7)x^{n+1}}{(n^2+3n+7)x^n}\right| = \frac{((n+1)^2+3(n+1)+7)}{(n^2+3n+7)}|x|$ What a mess! But its limit is just $|x|$ as $n\to\infty$ because the quotient is just two quadratic polynomials in n whose highest degree coefficients are both 1. Therefore the radius of convergence is 1. I did *not* ask for the "interval of convergence" and therefore you don't need to analyze what happens when $x=\pm 1$ (it diverges at both numbers).

b) Inside its radius of convergence, the sum of this series is a rational function (a quotient of polynomials). What is this function?

Comment: you may want to break up the series into several pieces and handle each piece separately. The answer is not necessarily "pretty".

Answer Start with $\sum_{n=1}^\infty x^n = \frac{x}{1-x}$ since this is a geometric series. Apply $\frac{d}{dx}$ to this equation, and get $\sum_{n=1}^\infty nx^{n-1} = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1}{(1-x)^2}$. Multiplying by x produces $\sum_{n=1}^\infty nx^n = \frac{x}{(1-x)^2}$. To get the n^2 we need to differentiate again (yes, this is a bit tedious). So $\sum_{n=1}^\infty n^2x^{n-1} = \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \frac{1(1-x)^2-2(1-x)(-1)x}{(1-x)^4} = \frac{(1-x)+2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}$ and multiply again by x to get the powers "right": $\sum_{n=1}^\infty n^2x^n = \frac{x+x^2}{(1-x)^3}$. Now add, adjusting with 3 and 7: the sum of the series is $\frac{x+x^2}{(1-x)^3} + 3\frac{x}{(1-x)^2} + 7\frac{x}{1-x} = \frac{x+x^2+3x(1-x)+7x(1-x)^2}{(1-x)^3}$, surely a quotient of polynomials. I see no reason to "simplify" the answer here.

(12) 9. True or false? If false, give an example to show that the implication is not true. If true, briefly explain why.

a) Suppose that all of the a_n's are positive. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges.

Answer True. If $\sum_{n=1}^\infty a_n$ converges, eventually the terms are less than 1. Then the corresponding terms of the series $\sum_{n=1}^\infty \frac{1}{a_n}$ will all be *greater* than 1. That series cannot converge since its terms do not approach 0.

b) Suppose that all of the a_n's are positive. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges.

Answer False. The harmonic series $\sum_{n=1}^\infty \frac{1}{n}$ diverges and all of its terms are positive. The series of its reciprocals is $\sum_{n=1}^\infty n$ which surely diverges. A colleague suggested a better and simpler example: take all the a_n's to be 1! b) then asserts that this series converges if it diverges: a rather oxymoronic (self-contradictory) mathematical joke!