Calculus problems for 640:192:03

Do two of these four problems, please.

1. Prove that $\{\sin(n)\}$ diverges. Please supply a proof: a convincing argument whose steps are *not* dependent upon numerical conjectures or guesses.
Hint: if you could show it doesn't get very close to any one number, then you're just about done. Is $\sin(n)$ "often" bigger than $\frac{1}{\sqrt{2}}$? Or "often" less than $-\frac{1}{\sqrt{2}}$? (How would this help you if you knew it?) How big and how small does the "whole" function, sine, get? How big is π compared to 1?

2. Consider the following sequences:

a) $a_n = \left(1+\left(\frac{1}{n}\right)\right)^n$ b) $b_n = \left(1+\left(\frac{1}{n^2}\right)\right)^n$ c) $c_n = \left(1+\left(\frac{1}{\sqrt{n}}\right)\right)^n$

Find the first five terms of each of these sequences. What can you say about the convergence of each of these sequences? (Why?)

3. Suppose α and β are positive numbers. Decide whether the following sequences converge. If they converge, try to find their limits.† (You must give reasons for your assertions!)

a) $d_n = \sqrt[n]{\alpha^n + \beta^n}$ b) $e_n = \sqrt[n]{\alpha^n + \beta}$

4. a) If you enter 5 in your 10 digit calculator, and then hit the square root button 20 times, you'll probably get:

$$1.00000\,15348$$

and if you then subtract 1 and multiply by 1,048,576 you'll get:

$$1.60943\,91475$$

but on the other hand, the same calculator will tell you that $\ln 5$ is:

$$1.60943\,79124$$

and I wonder, is this just a coincidence? **Well, is it?** Hint: 1,048,576 is 2^{20}.

b) Given a positive number, x, outline a strategy for computing $\ln x$ only with the "primitive" arithmetic operations ($+$, $\times$, $-$, $/$) and square root ($\sqrt{\ }$). Your strategy should involve asserting (and verifying) that a certain sequence which can be easily computed with the listed operations always converges to $\ln x$.

† Your answers may involve both numbers and their relationship.