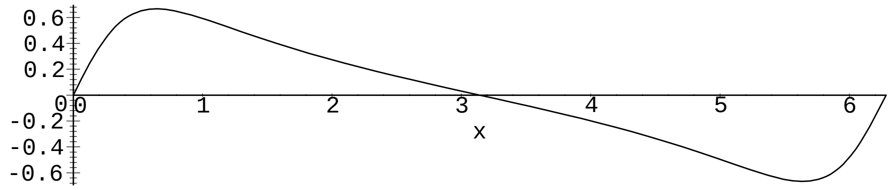
$F_{20}(x) = \sum_{n=1}^{20} \frac{1}{2^n} \sin(nx)$ plotted by `maple` on $[0, 2\pi]$. Note that $2^{-20} < \frac{1}{1{,}000{,}000}$.



Comment The function F can actually be written in "closed form" – that is, as a simple combination of familiar classical functions. The following was observed in conversation with Professor Amy Cohen and was *not* discussed in class.

We assume Euler's formula: $e^{ix} = \cos x + i \sin x$. Students in Math 192 were told of this formula and others obtained by extending to complex numbers various power series facts. Since the imaginary part of e^{inx} is $\sin nx$, we see that the imaginary part of the sum

$$\sum_{n=1}^{\infty} \frac{e^{inx}}{2^n}$$

is $F(x)$. But that sum is a geometric series whose first term is $\frac{e^{ix}}{2}$ and whose ratio is also $\frac{e^{ix}}{2}$. Therefore $F(x)$ is the imaginary part of $\left(\frac{1}{1-\frac{e^{ix}}{2}}\right) \cdot \left(\frac{e^{ix}}{2}\right)$. It is routine complex number arithmetic to write the imaginary part of this expression. First, multiply the fraction "top and bottom" by $1 - \frac{e^{-ix}}{2}$, the complex conjugate of the denominator. Then the bottom becomes $1 + \frac{1}{4} - \left(\frac{e^{ix}+e^{-ix}}{2}\right)$, and the last expression is just $\cos x$. Therefore the denominator is $\frac{5}{4} - \cos x$. The numerator is $\left(1 - \frac{e^{-ix}}{2}\right) \cdot \left(\frac{e^{ix}}{2}\right) = \frac{e^{ix}}{2} - \frac{1}{4} = \frac{1}{2}\left(\cos x + i \sin x - \frac{1}{2}\right)$. The whole expression is $\frac{\frac{1}{2}\left(\cos x + i \sin x - \frac{1}{2}\right)}{\frac{5}{4} - \cos x}$. The imaginary part is easy to see, so

$$F(x) = \frac{\frac{1}{2}\sin x}{\frac{5}{4} - \cos x} = \frac{2\sin x}{5 - 4\cos x}$$

Asking `maple` to sketch a graph of this formula over the interval $[0, 2\pi]$ produces a graph remarkably like what's above. To some extent I'm glad (!) that I *didn't* show students this argument. I insisted that knowing the graph of a partial sum to within $\pm 2^{-20}$ meant that we were really *seeing* the graph of F, because the drawn line was thicker than the uncertainty in the graph. The neat trick giving an explicit formula wouldn't work if the denominator were $2^n + 1$, for example, but the estimation techniques would still be useful.

The formula allows us to explore what seems to be the rather peculiar flatness of F's graph near π. `maple` says this is the initial segment of the Taylor series of F at $x = \pi$:

$$F(x) = -\frac{2}{9}(x-\pi) - \frac{1}{81}(x-\pi)^3 - \frac{7}{14580}(x-\pi)^5 + \frac{53}{1837080}(x-\pi)^7 + O\left((x-\pi)^9\right)$$

In fact, F really *isn't* peculiarly flat at π – compare the behavior of sinh and its Taylor series near $x = 0$. F has an inflection point at π just as sinh does at 0.