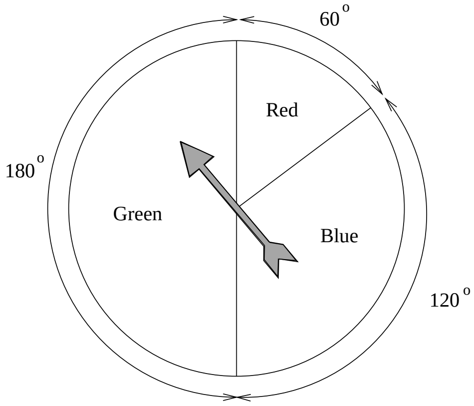
Payoffs

Suppose that we have a spinner as below:



What is the chance that an "honest" or "fair" spin lands on **Green**? I think it is $\frac{180°}{360°}$, or $\frac{1}{2}$. Similarly, we can easily see that the chance of **Blue** is $\frac{120°}{360°}$, or $\frac{1}{3}$, while the chance of **Red** is $\frac{1}{6}$. This is easy. So:

$$P(\mathbf{Green}) = \tfrac{1}{2} \qquad P(\mathbf{Blue}) = \tfrac{1}{3} \qquad P(\mathbf{Red}) = \tfrac{1}{6}$$

Here $P(\text{something})$ is the probability or chance that "something" happens. Clearly $0 \leq P(\text{something}) \leq 1$ and the sum of all the "somethings" that happen should be 1.

What may not be apparent is how much you *win* if each color pays a different amount. For example, suppose that

$$\begin{cases} \text{a } \mathbf{Green} \text{ spin pays off } \$30 \\ \text{a } \mathbf{Blue} \text{ spin pays off } \$15 \\ \text{a } \mathbf{Red} \text{ spin pays off } \$75 \end{cases}$$

Then how much is an "average" spin worth?

Phrased a bit differently, how much should someone be willing to *pay* to play this game? One naive answer might be: since there are three possible outcomes, and three possible rewards, then the average reward of a spin is the average of the three outcomes, or $\$\,\frac{30+15+75}{3} = \$40$. Some consideration of extreme cases might show that's too simple. If **Green** were worth $10,000 and the other two colors were worth nothing, then we'd expect about half of our spins to be **Green** in the long run, and about half the time to win $10,000 per spin, so that the *average* winning per spin would be $5,000. In our 30–15–75 payoff plan, we must compute a *weighted* average, and the weights are the chances, the probabilities, that each color will occur. So arrange things:

Outcome	**Probability**	**Payoff per spin**	**Expected winnings**
Green	$\frac{1}{2}$	$30	$15.00
Blue	$\frac{1}{3}$	$15	$5.00
Red	$\frac{1}{6}$	$75	$12.50

The total expected winnings (the *expectation*) will be $32.50, the correct weighted average of the probabilities and the payoffs. An entry "fee" of less than $32.50 would, in the long run, over many plays, yield a profit to the player. An entry fee greater than \$32.50 would, in the long run, over many plays, profit the "proprietor" of the spinner.

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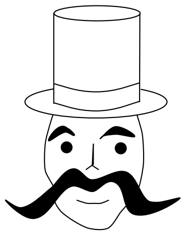
Let's try some real gambling. A gentleman just off the riverboat comes up to you, smiling. He carries a fair coin, with two different sides, one side heads, **H**, and one side tails, **T**. He offers to play a game with you. He will toss the coin. If **H** appears, he will pay you \$1. If **T** appears, he will toss the coin again. If, on the second toss, **H** appears, he will pay you \$2. If **T** appears again, he will toss another time. And so on ....



Your friendly gambler

In this case the "game" is complicated, and I think it needs to be specified more fully. The set of outcomes is the collection of coin tosses, **T T T . . . T H**. That is, for each non-negative integer, n, one possible outcome is $(n-1)$ **T**'s followed by an **H**. Let's call this $\mathcal{S}_n$. What's the probability of $\mathcal{S}_n$? We're asking for n straight specified tosses of a fair coin, so $P(\mathcal{S}_n)$ must be $\left(\frac{1}{2}\right)^n$. The gambler will pay n dollars if $\mathcal{S}_n$ occurs. Here are some questions to consider:

- What if the coin *never* lands heads?

  The probability of this happening seems to be $\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$. This may be an example of a conceivable (?) event which never happens (??).

- Would you pay the gambler, an honest, genial individual, 50 cents to play this game?

  Of course. You've got to win at least a dollar.

- Would you pay the gambler, an honest, genial individual, one dollar to play this game?

  Surely, for the same reason as the previous question.

- Would you pay the gambler, an honest, genial individual, one million dollars to play this game?

  The gambler asserts that there are many, many positive integers. There are only finitely many of these less than one million, and infinitely many which are more than one million. Thus (according to the gambler) there has infinitely *more* chances of paying more than one million than you have of losing one million. (!!!) So pay the million and play the game.

We can try to compute this game's average payoff (its "expectation") in a way that's similar to the spinner game. So:

Outcome	**Probability**	**Payoff per game**	**Expected winnings**
H	$\frac{1}{2}$	\$1	\$0.50
T H	$\frac{1}{4}$	\$2	\$0.50
T T H	$\frac{1}{8}$	\$3	\$0.375
. . . . .	...	...	...
$\mathcal{S}_n$	$\frac{1}{2^n}$	$\$n$	$\$\frac{n}{2^n}$
. . . . .	...	...	...

There's several observations we can make about this table. The sum of the probabilities of the various outcomes (an infinite list of outcomes which is abbreviated!) is

$$\sum_{n=1}^{\infty} P(\mathcal{S}_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Although we've got an "infinite series", we have the friendliest of such series, a geometric series. A geometric series is one whose terms are created by taking a first term, a, and multiplying that term repeatedly by one fixed number, r, the ratio, to create $a + ar + ar^2 + ar^3 + \ldots$. If S is the sum of this series, then

$$S = a + ar + ar^2 + ar^3 + \ldots = a + r(a + ar + ar^2 + \ldots) = a + rS$$

and this linear equation in S can be solved to get the formula $S = \frac{a}{1-r}$. Our specific geometric series has both a and r equal to $\frac{1}{2}$, so its sum is 1, as we should hope, since we've made a list of all (non-negligible) possible outcomes.

What about the average payoff? If the entry fee is *less* than the average payoff, in the long run, over repeated plays, we'd expect to profit, and the honest gambler tossing the coin for us would expect to lose money. If the entry fee is *more* than the average payoff, in the long run, over repeated plays, we'd expect to loose, and our honest, genial friend would expect to profit.

We need to look at the sum of the expected winnings:

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \ldots = \sum_{n=1}^{\infty} \frac{n}{2^n}$$

It is perfectly possible that this expectation *could* be infinite. To emphasize this, modify the game to one in which the gambler kindly offers to pay out 2^n dollars if the first head occurs on the n^{th} toss: e.g., outcome $\mathcal{S}_n$. Then every play of the game offers us a chance to win (on average) $\sum_{n=1}^{\infty} \frac{2^n}{2^n} = \infty$ dollars. This seems to be quite a lot! But what does it mean? There's certainly no one play or coin toss sequence which will win infinitely many dollars. It really means that there's no upper bound on the average amount of winnings in the changed game, and thus there's no fair entry fee that the gambler could charge – the genial gambler would always lose money in the long run.

Let's return to the original game, where the payoff is $\sum_{n=1}^{\infty} \frac{n}{2^n}$. Note, please, that this series is more complicated than the first. It is not a geometric series since the ratio between successive terms changes. For example, the ratio between the first and second terms is 1, and the ratio between the second and third terms is $\frac{3}{4}$. Let's see how to compute the payoff.

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Computing the payoff with calculus

Since we know (specializing the geometric series formula again, with a changed to 1 and r, to x):

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \ldots$$

We can differentiate this formula to get:

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + \ldots$$

Now multiply by x:

$$\frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + 4x^4 + \ldots$$

If we fix x to be $\frac{1}{2}$ we obtain:

$$2 = 0 + \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^4 + \ldots$$

Computing the payoff with algebra & magic

We can use the geometric series formula to write the following:

$$\begin{array}{rcllll}
1 & = & \frac{1}{2} + & \frac{1}{2^2} + & \frac{1}{2^3} + & \frac{1}{2^4} + \ldots \\
\frac{1}{2} & = & & \frac{1}{2^2} + & \frac{1}{2^3} + & \frac{1}{2^4} + \ldots \\
\frac{1}{2^2} & = & & & \frac{1}{2^3} + & \frac{1}{2^4} + \ldots \\
\frac{1}{2^3} & = & & & & \frac{1}{2^4} + \ldots \\
& \vdots & & & &
\end{array}$$

If we add this array *vertically* we'll get an equation whose left-hand side is just 2 (because it is just the sum of another geometric series with $a = 1$ and $r = \frac{1}{2}$), and whose right-hand side magically is $\frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^4 + \ldots$

In both cases the average payoff from the coin-flipping game is 2. So an entry fee of less than \$2 will result, over the long term, in profit for the player, while a fee of more than \$2 will give the gambler an edge.

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Those adept in calculus can apply the ratio test to see that $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges. Good applications of the ratio test can even give an estimate of the sum. But some sort of trickery must be used to compute the sum exactly.

Another game, and maybe some more magic

Suppose that we're offered a payoff of n^2 for the coin-flipping sequence, $\mathcal{S}_n$. For example, if the first head occurs on the third toss, we receive \$9. The average payoff for this game is then $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$. We can compute this by using almost the same tricks:

With calculus

Begin with

$$\frac{x}{(1-x)^2} = 0 + 1x + 2x^2 + 3x^3 + 4x^4 + \ldots$$

Differentiate to get:

$$\frac{1+x}{(1-x)^3} = 0 + 1 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots$$

Multiply by x:

$$\frac{x+x^2}{(1-x)^3} = 0 + 1x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \ldots$$

If we now fix x to be $\frac{1}{2}$ we see the formula:

$$6 = 0 + 1\frac{1}{2} + 2^2\left(\frac{1}{2}\right)^2 + 3^2\left(\frac{1}{2}\right)^3 + 4^2\left(\frac{1}{2}\right)^4 + \ldots$$

With algebra & magic

We know that

$$\text{Equation } \mathbf{A}: \quad 1 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \ldots$$

and

$$\text{Equation } \mathbf{B}: \quad 2 = \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + 4\left(\frac{1}{2}\right)^4 + \ldots = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \ldots$$

And now let's multiply by powers of $\frac{1}{2}$ and write the equations more agreeably:

$$\begin{array}{ccccccccccc}
\mathbf{A}: & 1 & = & \frac{1}{2} & + & \frac{1}{2^2} & + & \frac{1}{2^3} & + & \frac{1}{2^4} & + \ \ldots \\
\frac{1}{2}\,\mathbf{A}: & \frac{1}{2}\cdot 1 & = & & & \frac{1}{2^2} & + & \frac{1}{2^3} & + & \frac{1}{2^4} & + \ \ldots \\
\frac{2}{2}\,\mathbf{B}: & \frac{2}{2}\cdot 2 & = & & & \frac{2\cdot 1}{2^2} & + & \frac{2\cdot 2}{2^3} & + & \frac{2\cdot 3}{2^4} & + \ \ldots \\
\frac{1}{2^2}\,\mathbf{A}: & \frac{1}{2^2}\cdot 1 & = & & & & & \frac{1}{2^3} & + & \frac{1}{2^4} & + \ \ldots \\
\frac{2}{2^2}\,\mathbf{B}: & \frac{2}{2^2}\cdot 2 & = & & & & & \frac{2\cdot 1}{2^3} & + & \frac{2\cdot 2}{2^4} & + \ \ldots \\
 & & \vdots & & & & & & & &
\end{array}$$

Add the equations vertically. The right-hand sides of the equations add up to $\frac{1^2}{2} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \ldots$. *You* should write the two missing equations to complete the entries in the column for $\frac{1}{2^4}$, please. The collection of equations written isn't an accident, but exploits the fact that $(J+1)^2 - J^2 = 2J + 1$.

The left-hand sides become

$$1 + \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\right)1 + \left(\frac{2}{2} + \frac{2}{2^2} + \frac{2}{2^3} + \ldots\right)2 = 1 + 1\cdot 1 + 2\cdot 2 = 6$$

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Therefore the payoff for the n^2 game is 6. The fact that the sums of these infinite series are *integers* is somewhat unexpected. Even more amusing is that the payoff for an n^k game (with k any positive integer) must be an integer. A more direct version of this assertion is:

$$\text{If } k \text{ is an integer, then } \sum_{n=1}^{\infty} \frac{n^k}{2^n} \text{ is an integer.}$$

Why is this true? These sums can be computed in various ways. A clumsy inductive proof of the statement above can be given by using calculus to compute the sums. In outline, take the geometric series for $\frac{1}{1-x}$, then repeatedly differentiate and multiply by x, and then evaluate the resulting formula when $x = \frac{1}{2}$. Of course, any actual computations can be done with a computer algebra system such as `maple`. I did this, and got the payoffs for k between 0 and 4. They turn out to be 1, 2, 6, 26, and 150.

What may be a better way of seeing why these sums are integers is the magical approach. If k is a non-negative integer, define A_k to be $\sum_{n=1}^{\infty} \frac{n^k}{2^n}$. If $i \geq j$ and both i and j are non-negative integers, recall that $\binom{i}{j}$ is the binomial coefficient, $\frac{i!}{j!(i-j)!}$. Each binomial coefficient is a positive integer. Then for $k \geq 1$ the following recurrence is true:

$$A_k = 1 + \sum_{n=0}^{k-1} A_n \binom{k}{k-n}$$

Since (inductively!) all the terms on the right-hand side of this equation are positive integers, so is the result on the left. The equation is true because it just exploits the Binomial Theorem's expansion of $(J+1)^k - J^k$.

We know $A_0 = 1$ and $A_1 = 2$ and $A_2 = 6$. Let's compute A_3 and A_4:

$$A_3 = 1 + \sum_{n=0}^{2} A_n \binom{3}{3-n} = 1 + A_0\binom{3}{3} + A_1\binom{3}{2} + A_2\binom{3}{1} = 1 + 1\cdot 1 + 2\cdot 3 + 6\cdot 3 = 26$$

$$\begin{aligned} A_4 = 1 + \sum_{n=0}^{3} A_n \binom{4}{4-n} &= 1 + A_0\binom{4}{4} + A_1\binom{4}{3} + A_2\binom{4}{2} + A_3\binom{4}{1} \\ &= 1 + 1\cdot 1 + 2\cdot 4 + 6\cdot 6 + 26\cdot 4 = 150 \end{aligned}$$

Here is further information about the sequence $\{A_k\}$ and some related questions.

What (if anything) does the sequence of numbers $\{A_k\}$ count? If the sequence enumerated some objects, then we could immediately see that the A_k's are integers. I asked several combinatorists, and didn't get an answer. I consulted *The Encyclopedia of Integer Sequences* by Sloane and Plouffe (Academic Press, 1995), a wonderful source of wild information, and didn't find the sequence listed. But then I did what I should have done first: I asked Neil Sloane's program, `sequences@research.att.com`, about the sequence. The remainder of this paragraph should be read as an enthusiastic recommendation for this program. An e-mailed message consisting of the line `lookup 1 2 6 26 150` to `sequences@research.att.com` got a lengthy response. The program first searched its memory for matching sequences. If the search had not been successful, I could have directed a program to perform some transformations on the numbers supplied to see if any transformed sequences could have been matched. In this case, the response from `sequences` did identify the sequence as one already known (sequence A000629), and it gave the recursion obtained above. It listed $-\ln(2-e^x)$ as the exponential generating function of $\{A_k\}$: that is, the Taylor series centered at 0 of this function is $\sum_{k=1}^{\infty} \frac{A_k}{k!} x^k$. In addition, A_k turns out to be the closest integer to $\frac{k!}{(\ln 2)^{k+1}}$. The program also answered the initial question in this paragraph – there are some complicated combinatorial objects which the A_k's count. What a nice program!

Why do the payoffs discussed above turn out to be integers? There are other numbers (say, α) so that $B_k = \sum_{n=1}^{\infty} \frac{n^k}{\alpha^n}$ is always an integer when k is a non-negative integer. The sequence $\{B_k\}$ must satisfy the following recurrence when $k \geq 1$:

$$B_k = B_0 \left(1 + \sum_{n=0}^{k-1} B_n \binom{k}{k-n}\right)$$

Because of this recurrence, if B_0 is an integer, then *every* member of the sequence $\{B_k\}$ is an integer. Note that $\alpha = 1 + \frac{1}{B_0}$. The sequence of payoffs (called $\{A_k\}$) analyzed above resulted from the choice $B_0 = 1$ with $\alpha = 2$. If $B_0 = 2$ then $\alpha = \frac{3}{2}$, and the first B_k's are 2, 6, 30, 222, 2190, 27006, 399630, and 6899262. This sequence does *not* seem to be on Neil Sloane's list!