(17) 1. The two planes

$$x + 2y - 3z = 5$$

and

$$3x - y + z = 2$$

intersect in a straight line. Find parametric equations of this line.

Answer The ingredients for parametric equations for a line are a point the line goes through and a vector in the direction of the line.
For the point: we need a point on both planes. If we assume the point has x-coordinate 0 then we need to solve the following system of linear equations: $\begin{cases} 2y - 3z = & 5 \\ -y + z = & 2 \end{cases}$. We can double the second equation, getting $-2y + 2z = 4$ and add it to the first to obtain $-z = 9$ so $z = -9$. Now we can get y from, say, the equation $-y + z = 2$. Then $y = z - 2$ and $y = -11$. The point $(0, -11, -9)$ *is* on both planes (substitute!).
Comment There are *many* points on both planes.
For the vector: we need a vector in the direction of the line. The vector $\overrightarrow{v} = 1\overrightarrow{i} + 2\overrightarrow{j} - 3\overrightarrow{k}$ is perpendicular to the first plane, and the vector $\overrightarrow{w} = 3\overrightarrow{i} - 1\overrightarrow{j} + 1\overrightarrow{k}$ is perpendicular to the second plane. If we knew a vector perpendicular to *both* of these, that vector would be in the direction of the line. The cross-product seems to be the easiest way to find such a vector. So we compute $\overrightarrow{v} \times \overrightarrow{w} = \det \begin{pmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{pmatrix} =$
$-1\overrightarrow{i} - 10\overrightarrow{j} - 7\overrightarrow{k}$.
Comment There are *many* choices of vectors in the direction of the line.
And we combine the information given to get the desired parametric equations: $\begin{cases} x = & -1w + 0 \\ y = & -10w + -11 \\ z = & -7w + -9 \end{cases}$.
Comment There are *many* choices of letter for the parameter.

(12) 2. Find an equation for the plane normal to (that is, perpendicular to) the curve

$$\begin{cases} x = t^2 \\ y = 1 - t^3 \\ z = t \end{cases}$$

at the point where $t = 1$.

Answer The ingredients for the equation for a plane are a point on the plane and a vector normal to the plane.
For the point: if we "plug in" $t = 1$ into the parametric equations for the curve, we get $(1, 0, 1)$ which is a point on the plane.
Comment There are *many* points on ... $\leftarrow$ I will omit this sort of comment from now on.
For the vector: a vector normal to this plane is a vector tangent to the curve at $t = 1$. This is, of course, the key observation. Computing a tangent vector at $t = 1$ is easy: differentiate the position vector. The derivative of $t^2\overrightarrow{i} + (1 - t^3)\overrightarrow{j} + t\overrightarrow{k}$ is $2t\overrightarrow{i} + -3t^2\overrightarrow{j} + 1\overrightarrow{k}$. When $t = 1$ the velocity vector (which is a tangent vector) is $2\overrightarrow{i} - 3\overrightarrow{j} + 1\overrightarrow{k}$.
And we combine the information given to get an equation for the plane: $2(x-1) + -3(y-0) + 1(z-1) = 0$.

(18) 3. Suppose K is a constant and

$$F(x, y) = xe^{Kx^2y}$$

a) Compute $\dfrac{\partial F}{\partial x}$ and $\dfrac{\partial^2 F}{\partial x^2}$ and $\dfrac{\partial F}{\partial y}$.

Answer If we remember the chain rule and the product rule, these computations are straightforward*.

$$\begin{aligned} F_x &= e^{Kx^2y} + xe^{Kx^2y}2xy = e^{Kx^2y} + 2Kx^2ye^{Kx^2y} \\ F_{xx} &= e^{Kx^2y}2Kxy + 4Kxye^{Kx^2y} + 2Kx^2ye^{Kx^2y}2Kxy = 6Kxye^{Kx^2y} + 4K^2x^3y^2e^{Kx^2y} \\ F_y &= xe^{Kx^2y}Kx^2 = Kx^3e^{Kx^2y} \end{aligned}$$

b) There is one value of K so that $F(x,y)$ satisfies the equation

$$\frac{\partial^2 F}{\partial x^2} = 2y^2\frac{\partial F}{\partial y} + 3yF$$

for all values of x and y. Find that value of K and verify that $F(x,y)$ then does satisfy the equation.

Answer Here is what I expected or hoped would happen. People would translate the "general" equation above into a statement involving the rather specific form given for F.
That means:

$$\overbrace{6Kxye^{Kx^2y} + 4K^2x^3y^2e^{Kx^2y}}^{\frac{\partial^2 F}{\partial x^2}} = \overbrace{2y^2\left(Kx^3e^{Kx^2y}\right)}^{2y^2\frac{\partial F}{\partial y}} + \overbrace{3y\left(xe^{Kx^2y}\right)}^{3yF}$$

Now let's massage the equation above. One obvious step is to factor e^{Kx^2y} as much as possible (undistribute!):

$$e^{Kx^2y}\left(6Kxy + 4K^2x^3y^2\right) = e^{Kx^2y}\left(2y^2 \cdot Kx^3 + 3y \cdot x\right)$$

Note that the exponential term is never 0. Divide by it and get this equation:

$$6Kxy + 4K^2x^3y^2 = 2y^2 \cdot Kx^3 + 3y \cdot x$$

which must be true for all x and y. The only way this can happen is for $6K = 3$ and for $4K^2 = 2K$. It "happens" that these two equations have the same solution, $K = \frac{1}{2}$.

Comment I didn't want this problem to be impossible. After all, the difficult parts of the exam were supposed to be 4b and 6c. It is, though, really easy to make errors differentiating. I did this problem immediately after the exam during my office hour. I did it *wrong*! I must have differentiated in some screwy and incorrect way (with five people watching – I hypnotize into error with no intent!). Then I got two equations for K which had different solutions, and therefore there would have been no function of the given type which solved the given equation. I don't exactly know what I did wrong then, but what is above seems to be correct. Doing these computations correctly takes concentration, and writing a good problem takes some care.

Lest you think I make up things irrelevant to nature or reality or ... Here is an example of a famous "real" equation, the Korteweg-deVries equation:

$$\text{Equation } \mathbf{KdV} \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0$$

It models shallow water waves in a canal. $u(x,t)$ is the displacement of water surface from rest height at time t and position x along the canal. Some famous "standing wave" solutions were first observed in the 1840's and then were analyzed experimentally. Theoretical study of solutions of this equation has been been successful only during the last 20 years! The specific "standing wave" solutions which were discovered 150 years ago look like $A\left(\text{sech}(Bx - Ct)\right)^2$ where A,B and C are certain constants and (yes!) that's the square of the hyperbolic secant, which is defined by $\text{sech}(w) = \dfrac{1}{\cosh w} = \dfrac{2}{e^w - e^{-w}}$.

Joke Perhaps I should have given you such a solution to verify in **KdV** on this exam?

* But see the comment at the end of the problem!

(18) 4. Suppose $Q(x,y)$ is a differentiable function of two variables and that

$$A = Q(1,1) \qquad B = \frac{\partial Q}{\partial x}(1,1) \qquad C = \frac{\partial Q}{\partial y}(1,1)$$

$$D = \frac{\partial^2 Q}{\partial x^2}(1,1) \quad E = \frac{\partial^2 Q}{\partial x \partial y}(1,1) \quad F = \frac{\partial^2 Q}{\partial y^2}(1,1)$$

Suppose also that $g(t) = Q(t^2, t^3)$.

a) Compute $g'(t)$ using the chain rule. Then verify that $g'(1) = 2B + 3C$.

Answer We're going to need the chain rule in an appropriate form. The setting will be the following: Suppose $\spadesuit$ is a function of x and y while x and y are, in turn, both functions of t. Then $\spadesuit$ will be a function of t alone controlled indirectly through x and y. We can compute the derivative of $\spadesuit$ with respect to t in terms of derivatives of intermediate functions. Here's the relevant statement of the chain rule:

$$\frac{d\spadesuit}{dt} = \left(\frac{\partial \spadesuit}{\partial x}(x,y)\right)\frac{dx}{dt} + \left(\frac{\partial \spadesuit}{\partial y}(x,y)\right)\frac{dy}{dt}$$

We can be more specific in this situation, since we know exactly what x and y are in terms of t. In fact, $x = t^2$ and $y = t^3$.

$$\text{Equation } \clubsuit \qquad \frac{d\spadesuit}{dt} = \left(\frac{\partial \spadesuit}{\partial x}(x,y)\right)2t + \left(\frac{\partial \spadesuit}{\partial y}(x,y)\right)3t^2$$

I'll use this equation with $\spadesuit$ replaced by Q itself because $g(t)$ is $Q(t^2, t^3)$. Then we get

$$\frac{dg}{dt} = \left(\frac{\partial Q}{\partial x}(x,y)\right)2t + \left(\frac{\partial Q}{\partial y}(x,y)\right)3t^2$$

When $t = 1$, then $x = 1^2 = 1$ and $y = 1^3 = 1$. The equation becomes

$$g'(1) = \left(\frac{\partial Q}{\partial x}(1,1)\right)2\cdot 1 + \left(\frac{\partial Q}{\partial y}(1,1)\right)3\cdot 1^2 = 2B + 3C$$

as desired.

b) Use the first part of your answer to a) and the chain rule to compute $g''(t)$. Then verify that $g''(1) = 2B + 6C + 4D + 12E + 9F$.

Answer This is a more elaborate computation. We begin with the equation

$$\frac{dg}{dt} = \left(\frac{\partial Q}{\partial x}(x,y)\right)2t + \left(\frac{\partial Q}{\partial y}(x,y)\right)3t^2$$

and differentiate everything with respect to t. The left-hand side of the result is what we want. The right-hand side has a number of layers. First, it is a sum. Each part of the sum is a product. Each product will need to be considered carefully.
The first product is:

$$\frac{d}{dt}\left(\left(\frac{\partial Q}{\partial x}(x,y)\right)2t\right) = \left(\frac{d}{dt}\left(\frac{\partial Q}{\partial x}(x,y)\right)\right)2t + \left(\frac{\partial Q}{\partial x}(x,y)\right)\frac{d}{dt}(2t)$$

The tough term is $\frac{d}{dt}\left(\frac{\partial Q}{\partial x}(x,y)\right)$ which we will analyze with equation $\clubsuit$, with $\spadesuit$ replaced by $\frac{\partial Q}{\partial x}$.

$$\text{Equation } \clubsuit \text{ changed} \qquad \frac{d\frac{\partial Q}{\partial x}}{dt} = \left(\frac{\partial \frac{\partial Q}{\partial x}}{\partial x}(x,y)\right)2t + \left(\frac{\partial \frac{\partial Q}{\partial x}}{\partial y}(x,y)\right)3t^2$$

There's a more standard way to write this equation.

$$\text{Equation } \clubsuit \text{ rechanged} \qquad \frac{d\frac{\partial Q}{\partial x}}{dt} = \left(\frac{\partial^2 Q}{\partial x^2}(x,y)\right) 2t + \left(\frac{\partial^2 Q}{\partial x \partial y}(x,y)\right) 3t^2$$

Now I'll put this into what I called the "first product":

$$\frac{d}{dt}\left(\left(\frac{\partial Q}{\partial x}(x,y)\right) 2t\right) = \left(\left(\frac{\partial^2 Q}{\partial x^2}(x,y)\right) 2t + \left(\frac{\partial^2 Q}{\partial x \partial y}(x,y)\right) 3t^2\right) 2t + \left(\frac{\partial Q}{\partial x}(x,y)\right) \frac{d}{dt}(2t)$$

The right-hand side is

$$\left(\left(\frac{\partial^2 Q}{\partial x^2}(x,y)\right) 2t + \left(\frac{\partial^2 Q}{\partial x \partial y}(x,y)\right) 3t^2\right) 2t + \left(\frac{\partial Q}{\partial x}(x,y)\right) 2$$

because the derivative of $2t$ is 2.
Now let's handle the second product.

$$\frac{d}{dt}\left(\left(\frac{\partial Q}{\partial y}(x,y)\right) 3t^2\right) = \left(\frac{d}{dt}\left(\frac{\partial Q}{\partial y}(x,y)\right)\right) 3t^2 + \left(\frac{\partial Q}{\partial y}(x,y)\right) \frac{d}{dt}\left(3t^2\right)$$

We imitate what we did before, this time using equation $\clubsuit$ with $\spadesuit$ changed to $\dfrac{\partial Q}{\partial y}(x,y)$.

$$\text{Equation } \clubsuit \text{ newly changed} \qquad \frac{d\frac{\partial Q}{\partial y}}{dt} = \left(\frac{\partial \frac{\partial Q}{\partial y}}{\partial x}(x,y)\right) 2t + \left(\frac{\partial \frac{\partial Q}{\partial y}}{\partial y}(x,y)\right) 3t^2$$

Again let's rewrite:

$$\text{Equation } \clubsuit \text{ newly rechanged} \qquad \frac{d\frac{\partial Q}{\partial y}}{dt} = \left(\frac{\partial^2 Q}{\partial x \partial y}(x,y)\right) 2t + \left(\frac{\partial^2 Q}{\partial y^2}(x,y)\right) 3t^2$$

Now back:

$$\frac{d}{dt}\left(\left(\frac{\partial Q}{\partial y}(x,y)\right) 3t^2\right) = \left(\left(\frac{\partial^2 Q}{\partial x \partial y}(x,y)\right) 2t + \left(\frac{\partial^2 Q}{\partial y^2}(x,y)\right) 3t^2\right) 3t^2 + \left(\frac{\partial Q}{\partial y}(x,y)\right) \frac{d}{dt}\left(3t^2\right)$$

The right-hand side becomes

$$\left(\left(\frac{\partial^2 Q}{\partial x \partial y}(x,y)\right) 2t + \left(\frac{\partial^2 Q}{\partial y^2}(x,y)\right) 3t^2\right) 3t^2 + \left(\frac{\partial Q}{\partial y}(x,y)\right) 6t$$

because the derivative of $3t^2$ is $6t$.
We now put it all together:

$$\left(\frac{d}{dt}\right)\frac{dg}{dt} =$$

$$\left(\left(\frac{\partial^2 Q}{\partial x^2}(x,y)\right) 2t + \left(\frac{\partial^2 Q}{\partial x \partial y}(x,y)\right) 3t^2\right) 2t + \left(\frac{\partial Q}{\partial x}(x,y)\right) 2 + \left(\left(\frac{\partial^2 Q}{\partial x \partial y}(x,y)\right) 2t + \left(\frac{\partial^2 Q}{\partial y^2}(x,y)\right) 3t^2\right) 3t^2 + \left(\frac{\partial Q}{\partial y}(x,y)\right) 6t$$

That's some formula! We will evaluate it when $t = 1$ and $x = 1$ and $y = 1$. We get:

$$g''(1) =$$

$$\left(\left(\frac{\partial^2 Q}{\partial x^2}(1,1)\right) 2 \cdot 1 + \left(\frac{\partial^2 Q}{\partial x \partial y}(1,1)\right) 3 \cdot 1^2\right) 2 \cdot 1 + \left(\frac{\partial Q}{\partial x}(1,1)\right) 2 + \left(\left(\frac{\partial^2 Q}{\partial x \partial y}(1,1)\right) 2 \cdot 1 + \left(\frac{\partial^2 Q}{\partial y^2}(1,1)\right) 3 \cdot 1^2\right) 3 \cdot 1^2 + \left(\frac{\partial Q}{\partial y}(1,1)\right) 6 \cdot 1$$

Look up these numbers in the "dictionary" given at the start of the problem. The result is what's desired.

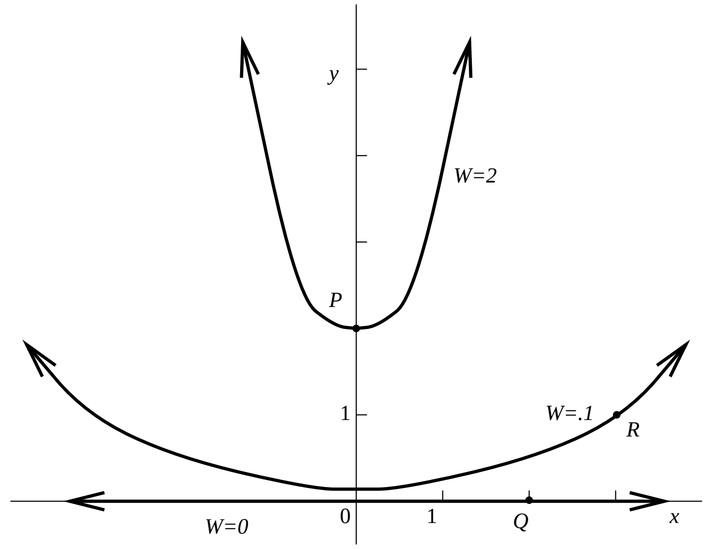
$$g''(1) = (D \cdot 2 + E \cdot 3)2 + B \cdot 2 + (E \cdot 2 + F \cdot 3)3 + C \cdot 6 = 2B + 6C + 4D + 12E + 9F$$

(15) 5. Sketch the three level curves of the function

$$W(x,y)=\frac{y}{x^2+1}$$

which pass through the points $P=(0,2)$ and $Q=(2,0)$ and $R=(3,1)$. Be sure to label each curve with the appropriate function value and be sure that your drawing is clear and unambiguous.

Answer If $\frac{y}{x^2+1}=C$ then $y=C(x^2+1)=Cx^2+C$. These are parabolas (o.k.: if you want to disagree about what happens when $C=0$ – it is a "degenerate parabola", a straight line). Anyway, when $C>0$ the parabolas open up, and have their vertex at $(C,0)$. The "spread" of the parabolas is also controlled by C – the larger C is, the "narrower" the parabolas are. If $C<0$, the graphs flip over the x-axis; these parabolas open down.
Since $W(0,2)=2$, the contour line/level curve through P is the parabola $y=2x^2+2$.
Since $W(2,0)=0$, the contour line/level curve through Q is the "parabola" or straight line $y=0$, the x-axis.
Since $W(3,1)=.1$, the contour line/level curve through R is the parabola $y=.1x^2+.1$, lower and wider (?) than the other parabola sketched.



(20) 6. Suppose the curve C has position vector

$$\vec{r}(t)=\cos t\,\vec{i}+\sin t\,\vec{j}+t^2\,\vec{k}$$

a) What is the unit tangent vector to C when $t=\pi$?

Answer $\vec{r}\,'(t)=-\sin t\,\vec{i}+\cos t\,\vec{j}+2t\,\vec{k}$. When $t=\pi$, this is $0\,\vec{i}-\vec{j}+2\pi\,\vec{k}$. The velocity vector is tangent to the curve. Make it unit length by dividing by its magnitude. The magnitude is $\sqrt{1+(2\pi)^2}$ so the desired vector is $\frac{-1}{\sqrt{1+(2\pi)^2}}\vec{j}+\frac{2\pi}{\sqrt{1+(2\pi)^2}}\vec{k}$. I see **no** useful "simplification".

b) Write an integral for the length of the part of C between $t=0$ and $t=2\pi$.
DO NOT EVALUATE THE INTEGRAL.

Answer The integral of the magnitude of the velocity vector ("speed") gives the length of the curve. So the answer is (using the computation done above) $\int_0^{2\pi}\sqrt{(\sin t)^2+(\cos t)^2+(2t)^2}\,dt$. It may be almost painful to leave this untouched but this answer is **fine**. You don't need to notice anything!

c) The curvature of this curve is given by the formula

$$\kappa(t)=\sqrt{\frac{5+4t^2}{(4t^2+1)^3}}$$

which you should **not** verify. Assume it is correct. Even though the first two components of this curve describe uniform circular motion, $\lim_{t\to\infty}\kappa(t)=0$. Explain briefly why this can happen, using complete English sentences possibly assisted by properly labelled diagrams.

Answer Motion in x and y is certainly uniform circular motion, but the stretching in z is **non-linear**, and increases rapidly as t increases. A few pictures from `maple` may help. These pictures may take some study to be adequately understood. Figure I illustrates this curve for t in the interval $[0, 8\pi]$. The curve makes 4 "loops" corresponding to the 4 multiples of 2π, and z increases from 0 to $(8\pi)^2 \approx 632$. Figure II shows the curve for t in the interval $[10\pi, 18\pi]$. Again, the curve makes 4 loops because the change in t, what I call $\triangle t$, is 8π. Notice that this picture doesn't seem to differ that much from Figure I. The reason (which may be well-known to some of you by now) is that `maple` "autoscales". That is, the program is instructed to have images occupy as much of the screen as possible*. This can lead to deceptive pictures! Look very closely at the z scale in these pictures (if you can see it). For example, in Figure II, z goes from $(10\pi)^2 \approx 987$ to $(18\pi)^2 \approx 3200$ – so the vertical scale for 4 loops changes from a distance of 632 in Figure I to a distance of more than 2200 in Figure II! If we wish to see the curve starting at $t = 20\pi$ in a box where the vertical change is about 600 as in Figure I, consider the part of the curve from $20\pi \approx 3950$ to $20\pi + 4 \approx 4470$. This is shown in Figure III. Now $\triangle z \approx 520$ although $\triangle t$ is just 4: there's the non-linearity. The curve travels almost the same vertical distance that it did in the 8 turns in Figure I in less than $\frac{4}{2\pi} \approx .6$ turns in Figure III. The curve is *much* flatter because it rises so much that the "turning" of x and y is mostly cancelled. The curvature is much smaller for larger t. Curvature is part of the geometry of the curve, and speed of travel along the curve is irrelevant. Speed on the curve with the parameterization given is much greater in Figure III than in Figure I or II but speed doesn't matter to curvature.

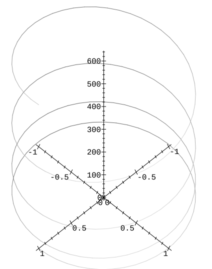


Figure I

Starting t: 0
$\triangle t$: $8\pi \approx 25$
Starting z: 0
$\triangle z$: $(8\pi)^2 \approx 632$

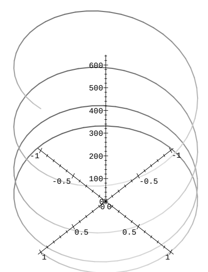


Figure II

Starting t: $10\pi \approx 31$
$\triangle t$: $8\pi \approx 25$
Starting z: $(10\pi)^2 \approx 987$
$\triangle z$: ≈ 2210

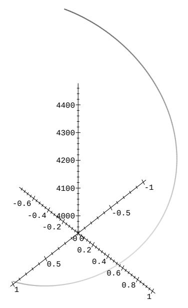


Figure III

Starting t: $20\pi \approx 63$
$\triangle t$: 4
Starting z: $(20\pi)^2 \approx 3948$
$\triangle z$: ≈ 520

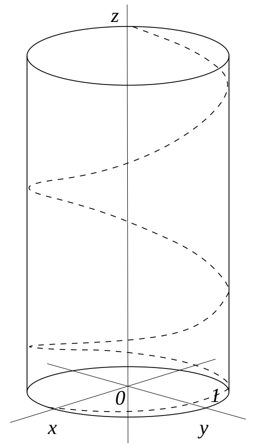
I hope what's written above explains the gradual flattening of the curve which the asymptotic behavior of the curvature predicts. Below on the right is a more artistic non-`maple` picture of this curve. The curve is on the circular cylinder $x^2 + y^2 = 1$ in (x, y, z)-space. It is *not* on a cone. The equation $x^2 + y^2 = 1$ with only two variables **does** describe a surface in space.

Comment I've now graded the problem. Perhaps question 6c should have been worded more specifically to get a reply whose content related directly to the geometry of the curve. I did **not** want a discussion of the limit of the function $\kappa(t)$ using first-semester calculus. Maybe the sentence beginning "Explain briefly ..." should've been: "Explain briefly why this happens using properties of the geometry of the curve. Write complete English sentences ..." Here is a possible student answer which would be entirely acceptable:

> The curve gets flatter as t increases because $z = t^2$ makes the loops spread out in the z direction. Therefore the effect of the circular motion decreases and the curve gradually gets flatter and flatter.

What do you think would happen if $z = \sqrt{t}$ instead of $z = t^2$?

Thanks for reading all this.



* The nice thing about programs is that they do what you tell them to do. The infuriating thing about programs is that they do what you tell them to do (which is not necessarily what you *want* them to do).