November 9, 1996

A sphere is a point
gravitationally

Probably the most important scientific text ever published was Newton's *Principia**. In the *Principia* Newton used mathematical methods to deduce a complete description of the universe from a small collection of assumptions ("laws"). The acceptance of his work and the enthusiasm it created for science in some parts of European society, accompanied by the success of science-based attempts to change nature, were essential ingredients in the later acceptance of science as an almost secular religion in Western civilization. Some historians of science believe that the difficulty of the specific problem discussed here noticeably delayed Newton's publication of his work.

We assume the "law of universal gravitation": two particles attract each other with a force directed between them whose magnitude is directly proportional to the product of the masses of the particles and is inversely proportional to the square of the distance between the particles. Consider a homogeneous sphere: a ball with constant density. How does it attract an external particle? Often students are told to assume the mass of the ball is concentrated at its center. So objects like the earth and the sun "become" point masses, substantially simplifying analysis of their motion under the influence of gravity. Calculus then can be used to verify results like Kepler's laws. Sequences of reasoning like this (from the "abstract" law of universal gravitation to observationally verifiable statements such as elliptical orbits of planets) enchanted readers of the *Principia*.

It is not obvious that the mass of a sphere appears externally to be concentrated at its center. This result relies on both the "inverse square" nature of the "law" and also on the geometry of the sphere. We'll see here that this result is correct. Physicists and mathematicians worked hard in the century following Newton to make these computations easier to understand. They succeeded, but the cost to the reader is learning even more new terminology and ideas. What's below is a rather naked approach to the computation.

Suppose the external mass, m, is located at a point Q whose distance to the center of a homogeneous sphere of radius R is D, with $D > R$. Our coordinate system will put the center of the sphere at the origin and will place Q on the z axis. Look at the picture on the next page. Consider a small piece of the sphere's mass located at a point P somewhere inside the sphere. The spherical coordinates ρ and θ and ϕ of P are of course subject to the restrictions $0 \leq \rho \leq R$ and $0 \leq \theta \leq 2\pi$ and $0 \leq \rho \leq \pi$. μ will be the sphere's density.

What's the magnitude of the force between P and Q? It is directly proportional to the product of the masses and inversely proportional to the square of the distance between the masses. The constant of proportionality is usually called G. If d is the distance from P to Q, then the magnitude of the force is

$$G\frac{m\,\overbrace{\mu\,dv}^{\text{the mass at } P}}{d^2}$$

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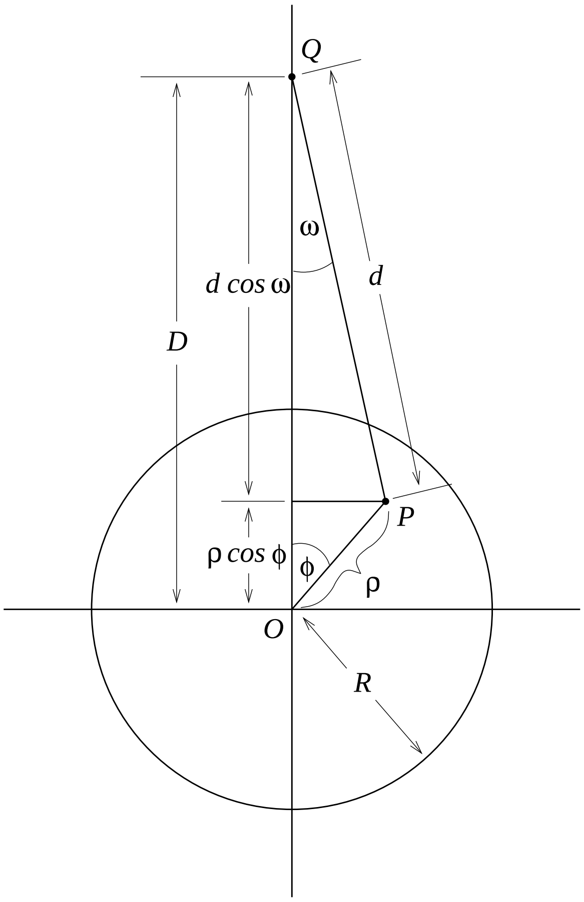
* *Philosophiae Naturalis Principia Mathematica* = *Mathematical Principles of Natural Philosophy*, published in 1687.

The force is directed from Q to P. We want the resultant (the sum) of all the forces for every point P inside the sphere. The $\vec{\imath}$ and $\vec{\jmath}$ components of the resultant are 0 because the sphere is symmetric with respect to the z axis, so each point P with rectangular coordinates (a, b, c) has a mirror image point $(-a, -b, c)$ in the sphere which pulls (only in the first two coordinates!) exactly opposite to P's pull. However, the pulls of these two "mirror" points reinforce each other in $\vec{k}$ direction. To get the vertical component of the QP force, multiply the magnitude above by $\cos\omega$, where ω is the angle between the line segment $\overline{QO}$ (lying on the z axis) and the line segment $\overline{QP}$. (Again, examine the picture!). To get the total force, we should add all of the P forces, for every point P inside the sphere. Therefore the quantity we want to compute is:



$$\underset{\text{the entire sphere}}{\iiint} \frac{Gm\mu\cos\omega}{d^2}\,dv$$

We've already seen what the limits for this triple integral are in spherical cooordinates. We need to write everything in terms of ρ, ϕ and θ. The picture will help. Newton's published proof of the conclusion we'll get used very elaborate geometric arguments, not calculus. There's evidence Newton did many computations with calculus and then translated his reasoning into geometric language, which was more acceptable to the scholars of his time. But the result we're discussing is definitely not easy using any approach.

What's $\cos\omega$? Since $D = d\cos\omega + \rho\cos\phi$, we know

$$\cos\omega = \frac{D - \rho\cos\phi}{d}$$

Now the quantity we want to compute is changed:

$$\underset{\text{the entire sphere}}{\iiint} \frac{Gm\mu\cos\omega}{d^2}\,dv = \underset{\text{the entire sphere}}{\iiint} \frac{Gm\mu\,(D - \rho\cos\phi)}{d^3}\,dv$$

The law of cosines applied to $\triangle QOP$ describes d in terms of D, ρ, and ϕ.

$$d^2 = D^2 + \rho^2 - 2\rho D\cos\phi$$

What we want to compute changes again:

$$\underset{\text{the entire sphere}}{\iiint} \frac{Gm\mu\,(D - \rho\cos\phi)}{d^3}\,dv = \underset{\text{the entire sphere}}{\iiint} \frac{Gm\mu\,(D - \rho\cos\phi)}{(D^2 + \rho^2 - 2\rho D\cos\phi)^{3/2}}\,dv$$

We write the integral in spherical coordinates, getting the last reformulation of our quantity before we actually evaluate it:

$$\underset{\text{the entire sphere}}{\iiint} \frac{Gm\mu\,(D-\rho\cos\phi)}{\left(D^2+\rho^2-2\rho D\cos\phi\right)^{3/2}}\,dv = \int_0^{2\pi}\int_0^R\int_0^\pi \frac{Gm\mu\,(D-\rho\cos\phi)}{\left(D^2+\rho^2-2\rho D\cos\phi\right)^{3/2}}\,\rho^2\sin\phi\,d\phi\,d\rho\,d\theta$$

The innermost integral of this formidable mess is the hardest:

$$\int_0^\pi \frac{Gm\mu\,(D-\rho\cos\phi)}{\left(D^2+\rho^2-2\rho D\cos\phi\right)^{3/2}}\,\rho^2\sin\phi\,d\phi$$

$Gm\mu\rho^2$ is a constant for this integration. So the integral to compute is

$$\int_0^\pi \frac{(D-\rho\cos\phi)}{\left(D^2+\rho^2-2\rho D\cos\phi\right)^{3/2}}\,\sin\phi\,d\phi$$

This integral is difficult. `maple` finds it irritating: evaluation took about 4 seconds with the implementation I'm currently using!* The method for computing it will be indirect. First, integrate by parts. The parts are:

$$u = D-\rho\cos\phi \quad\text{and}\quad dv = \frac{\sin\phi}{\left(D^2+\rho^2-2\rho D\cos\phi\right)^{3/2}}\,d\phi$$

and because $\sin\phi$ is on top of dv, we can antidifferentiate without too much difficulty:

$$du = \rho\sin\phi\,d\phi \quad\text{and}\quad v = \frac{-1}{\rho D\sqrt{D^2+\rho^2-2\rho D\cos\phi}}$$

Since $\int_a^b u\,dv = uv\big]_a^b - \int_a^b v\,du$ we need to look at two terms on the right-hand side of the equation. Here is the first term:

$$(D-\rho\cos\phi)\cdot\frac{-1}{\rho D\sqrt{D^2+\rho^2-2\rho D\cos\phi}}\Bigg]_0^\pi = \overbrace{\left(\frac{-(D-\rho(-1))}{\rho D\sqrt{D^2+\rho^2+2\rho D}}\right)}^{\phi=\pi} - \overbrace{\left(\frac{-(D-\rho(+1))}{\rho D\sqrt{D^2+\rho^2-2\rho D}}\right)}^{\phi=0}$$

$$= \left(\frac{-D-\rho}{\rho D\sqrt{(D+\rho)^2}}\right) - \left(\frac{-D+\rho}{\rho D\sqrt{(D-\rho)^2}}\right) = \frac{-1}{\rho D} - \frac{-1}{\rho D} = 0 \quad \text{Clearly (?) zero!}$$

We can handle the $-\int_a^b v\,du$ term by first canceling ρ's and $-$'s and then integrating, with the $\sin\phi$ on top again making antidifferentiation straightforward:

$$-\int_0^\pi \frac{1}{\rho D\sqrt{D^2+\rho^2-2\rho D\cos\phi}}\cdot(-\rho\sin\phi)\,d\phi = \int_0^\pi \frac{\sin\phi}{D\sqrt{D^2+\rho^2-2\rho D\cos\phi}}\,d\phi$$

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* Hmmm .... what if *Newton* had a computer?!

$$= \frac{\sqrt{D^2+\rho^2-2\rho D\cos\phi}}{D^2\rho}\Bigg]_0^\pi = \overbrace{\left(\frac{\sqrt{D^2+\rho^2+2\rho D}}{D^2\rho}\right)}^{\phi=\pi} - \overbrace{\left(\frac{\sqrt{D^2+\rho^2-2\rho D}}{D^2\rho}\right)}^{\phi=0}$$

$$= \frac{\sqrt{(D+\rho)^2}}{D^2\rho} - \frac{\sqrt{(D-\rho)^2}}{D^2\rho} = \frac{(D+\rho)}{D^2\rho} - \frac{(D-\rho)}{D^2\rho} = \frac{2}{D^2}$$

We assemble all the pieces for the next integral. We must include the "constants" that were temporarily discarded.

$$\int_0^R Gm\mu\rho^2\cdot\frac{2}{D^2}\,d\rho = \int_0^R \frac{2Gm\mu}{D^2}\,\rho^2\,d\rho = \frac{2Gm\mu}{D^2}\frac{\rho^3}{3}\Bigg]_0^R = \frac{2Gm\mu}{D^2}\frac{R^3}{3}$$

The last $d\theta$ integral just multiplies this result by 2π so our final answer is the following (with algebraic pieces arranged sensibly):

$$G\,\frac{m\left(\frac{4}{3}\pi R^3\mu\right)}{D^2}$$

Please recognize the expression in parentheses. It is the volume of a sphere of radius R multiplied by the constant density μ: precisely the mass of the homogeneous sphere. Of course D is the distance from the point mass m at Q to the center of the sphere. So we do have the same force between the point mass and the sphere as if the sphere's mass were "concentrated" at its center.

We've proved Newton's result.

Problem *The Flat Earth* Suppose the earth is flat. By that I mean we think of it as a two-dimensional infinite planar lamina (this is a word used in textbooks and defined by dictionaries to mean "a thin plate ... e.g., of stratified rock"). If Newton's law of universal gravitation is correct, show that the force of attraction between the infinite plate and an external point mass does *not* depend on the distance from the point mass to the plate.

Hints, Comments This computation can be set up using techniques similar to what was done here for the sphere. Put the point mass at $(0,0,D)$ and let the "flat earth" be the (x,y) plane. Write a *double* improper integral for the attraction between the plane and the external mass. Then the integral can be evaluated directly with polar coordinates.

If we think of electrical force (which is also an "inverse square" force) rather than gravity, this computation approximates the force on an external electron given by a charged plate which might be part of a capacitor. Certainly the capacitor's plate isn't infinitely large, but electrons are very small compared to a rectangular plate a few inches wide. The infinite planar approximation isn't far off and the result is useful.

The "inverse square" nature of these forces has been tested many times to quite good accuracy. The quadratic assumption seems to be correct.

Problem *Journey to the Center of the Earth* Suppose the earth is a homogeneous ball. What is the attraction of the earth on a point mass located *inside* the earth?

Hint Set up the problem using methods similar to what's done here for an external point mass. Break up the resulting triple integral into two pieces and analyze them separately. You'll see as you compute that the attraction of a thin homogeneous spherical shell on an *internal* point mass is 0 (!)*: be careful of $\sqrt{(D-\rho)^2}$ when $\rho>D$.

You know two special cases to check your final answer. If the point mass is in the center of the ball, the attraction (because of symmetry) should be 0. If the point mass is on the surface of the ball, the attraction is covered by the computation we did when $D=R$. The way your final answer varies with D (the distance from the point mass to the center of the ball, so $0\leq D\leq R$) should be quite simple.

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* **Joke** Thus the universe *could* be inside a beachball with diameter 10^{100} lightyears: no gravity would be felt!