AAAARRRRGGHHHH!!!

My apologies if you were in today's Math 251 lecture. Certainly the material I talked about is one of the most important topics of the course, probably the single MOST important topic of the first third of the course, and I messed it up. I have only apologies with little explanation. I was neither hung over nor inebriated, and had no more than my usual mental incapacities. I had even thought carefully (I believed!) about what I wanted to say. Sigh!

Below I'll describe for you approximately what I wanted to tell you. This is important for several reasons:

1. As above, one of the chief topics of the course.

2. Even though the material is NOT on the Friday exam, its analysis is very relevant to most of what we've done up to now: it really is a payoff for lots of irritating details with vectors, and a neat synthesis of partial derivatives and vector algebra. In particular, "legal" topics for Friday's ordeal include the chain rule, dot products, and level sets (curves, surfaces), all of which were involved in today's lecture.

Anyway, think of this as a field test of "distance learning", which is a really hot topic now. Sigh. I could be writing this from ALaska, and maybe I should be! If I did it right the first time, this e-mail message wouldn't be necessary. Really, my malfunctioning brain has not gotten better: I'm now afraid to write "2+3" because I think I might write "=7" after it. I hope this was the worst lecture of the semester!

What did I want to talk about? Let me outline it:

I wanted to investigate the rate of change of a function T(x,y,z) as I "moved" along a curve r(t)=<x(t),y(t),z(t)>. The chain rule applied to give me the following result (this is d/dt of T(x(t),y(t),z(t))):

        (T_x) (dx/dt) + (T_y) (dy/dt) + (T_z) (dz/dt).

Here T_x means "T-sub-x", the partial derivative of T with respect to x, etc.

The first interesting thing to notice is that this can be rewritten as a dot product of two vectors. One vector is <dx/dt, dy/dt, dz/dt>, which is the velocity vector of the curve. The other is the vector

                    T_x i + T_y j + T_z k

which is called the gradient of T. Because I can't make math characters in this ascii environment, I'll call this vector "grad T".

Writing the stuff up there as (grad T) dotted with (velocity) separates the effect of the curve from the effect of the function, which is really useful.

The speed of the function can be factored out of the expression (grad T) dot (velocity) so that the expression can be written as

(speed) times (grad T) dot (unit vector in direction of velocity)

Example: $T(x,y,z)=x^2 + 3yz$

gets you

$$\text{grad } T = 2x\ i + 3z\ j + 3y\ k$$

If you have a curve with position vector

$$r(t)= t^2\ i + t\ j + t^3\ k,$$

then when $t=2$, $r(2)$ is $4\ i + 2\ j + 8\ k$.

What's the "rate of change" of T on the curve r when $t=2$?

O.k., when $t=2$, then $x=2$, $y=2$, $z=8$, and then

$$\text{grad } T = (4\ i + 24\ j + 6\ k)$$

The velocity vector is

$$v(t) = 2t\ i + j + 3t^2\ k$$

so $v(2) = 4\ i + j + 12\ k$.

The rate of change of T along the curve when $t=2$ is therefore the dot product

$$(4\ i + 24\ j + 6\ k) \text{ dot } (4\ i + j + 12\ k) = 16 + 24 + 72 = 112$$

(I hope I get the arithmetic correct!!!)

Anyway, if the particle whose motion was described by $r(t)$ were moving twice as fast, then the rate of change would be doubled to 224. Or if the particle were moving "backwards" the rate of change would be -112.

The speed of movement along the curve just multiplies the rate of change. What's important is the direction of the velocity vector. The "direction" is defined to be a unit vector (a vector of length 1) which is a positive scalar multiple of the vector.

So to get the direction of

$$v(2) = 4\ i + j + 12\ k$$

we divide by $\text{sqrt}(4^2 +1^2 + (12)^2)$. This is a unit vector.

I'll call unit vectors, u.

The result of computing

$$(\text{grad } T) \text{ dot } u$$

can be thought of as the rate of change of T while moving in the direction u at "unit speed". This is called the "directional derivative of T in the direction u"

In the example above, grad T = (4 i + 24 j + 6 k), then if u is the unit vector i, the directional derivative is (4 i + 24 j + 6 k) dot i which is 4, the same as T_x (the rate of change of T with respect to x). If u = -j then the directional derivative is (4 i + 24 j + 6 k) dot (-j) which is -24, minus the value of T_y because we'd be moving "backwards" along the y direction from the point (4,2,8) and trying to see what the change was.

How can you "maximize" the directional derivative? That is, how can you select u, a unit vector, so that moving in the u direction will increase T "the most"? In the example above, if we chose u to be j, the increase in T would be 24. Is there any way of choosing a better direction? (Answer: yes, described below.)

This question is NOT abstract. Very frequently in applications small changes are allowed to be made to inputs, with the idea being to increase the value of a function. If the function depends on more than one variable, then what follows is the strategy used to select the direction of the changes.

What we do is realize that

$$(\text{grad } T) \text{ dot } u = |\text{grad } T| \; |u| \cos \theta$$

where theta is the angle between u and grad T. This is the usual dot product formula. Since u is a unit vector, |u| = 1 and we can maximize the directional derivative by arranging for cos theta to be 1 which happens when theta is 0. So the angle "between" grad T and u should be 0, so u should be a scalar multiple of grad T. But a unit vector in the direction of grad T is

$$\frac{1}{|\text{ grad } T \,|} \text{ grad } T$$

and that's the direction which will maximize the rate of change of T (!). And the maximum rate of change (the largest directional derivative) will actually be

$$\frac{1}{|\text{ grad } T \,|} \text{ grad } T \text{ dotted with grad } T$$

and since grad T dotted with grad T gives you $|\text{grad } T|^2$, one power cancels out so the result is just |grad T|.

In the example above, where grad T = (4 i + 24 j + 6 k), if I chose the unit vector to be

$$(4/\mathrm{sqrt}(4^2 + (24)^2+ 6^2))\ i + (24/\mathrm{sqrt}(4^2 + (24)^2+ 6^2))\ j + (6/\mathrm{sqrt}(4^2 + (24)^2+ 6^2))\ k$$

(WOW!) then I'll get the LARGEST directional derivative. And the value of that "change in T" will be $\mathrm{sqrt}(4^2 + (24)^2+ 6^2)$.

More decimally, to get the greatest change in T at the point (4,2,8) you should move in the direction

$$.15961\ i + .95770\ j + .23942\ k$$

and the value of the change will be

$$25.05991$$

This is not at all obvious!!! NOT OBVIOUS AT ALL, darn it!!!

There is even more. $T(x,y,z)=x^2 + 3yz$. Since $T(4,2,8) = 4^2 + 3(2)(8) = 64$ (I hope!) the point (4,2,8) must be a point on the level surface defined by the equation

$$x^2 + 3yz = 64$$

Moving along the level surface in directions tangent to the surface won't change the value of T very much at all. That means tangent directions are perpendicular to the gradient vector. (There's more on this in the book, of course). So since grad T = (4 i + 24 j + 6 k), this vector will be normal (perpendicular) to the level surface $T(x,y,z) = 64$ at the point (4,2,8). That means

$$4\ (x-4) + 24\ (y-2) + 6\ (z-8) = 0$$

is an equation of the plane tangent to $x^2 + 3yz = 64$ at (4,2,8).

Hey, I can't even easily draw the surface, but I CAN very easily write an equation of the tangent plane.

In two dimensions, the gradient vector is perpendicular to level CURVES of the surface. In hiking if you know how to use a contour map, you know to look for gradient vectors if you want the quickest (not necessarily easiest!) path up or down: hike perpendicular to the level curves. Several numerical techniques for getting max's and min's in several variables work by hiking up (or down) the gradient vectors.

Anyway, that is what I did want to say today. I ran overtime partially because my K-mart $3.00 pocket watch malfunctioned and partially because my brain didn't work well. Again, apologies. See you Thursday!