Mathematics 251: Lab 1 CALCULUS OF VECTOR FUNCTIONS

In this lab, we shall use Maple to compute dot and cross products of vectors, the length of a vector, the derivative of a vector function, and to plot vector functions. We shall also use Maple to compute and understand the curvature of a space curve.

Please turn in only the printout of your Maple worksheet. Include explicit answers to all questions asked, using the **text** feature of Maple to insert them in the worksheet (don't write them by hand). Insert all graphs directly into your worksheet; instructions were given in the initial handout. Remove from the worksheet any extraneous material such as the output of commands like `with(plots);` and any errors you have made.

In order to use Maple's commands for computing dot and cross products, it is first necessary to type the command `with(linalg):`. Type this command and also the command `with(plots):` which allows Maple to access various plotting commands.

1a. Execute the following sequence of commands which illustrate how vectors are defined in Maple, how Maple adds vectors, computes dot and cross products and the length of a vector, and forms the product of a scalar with a vector.

```
# Dot and Cross Products and Norm
with(linalg):  with(plots):
v:= vector([1,2,3]);
w:= vector([-1,1,2]);
dpvw:= dotprod(v,w);
cpvw:= crossprod(v,w);
nv:= norm(v,2);   # computes the length of v
nrmv:= sqrt(dotprod(v,v));  # another way to compute the length of v
z:= add(v,w);
u:= scalarmul(v,1/nrmv);  # produces a unit vector in the direction of v
```

1b. Use the `add` and `scalarmul` commands to have Maple compute the vector $\mathbf{v} - \mathbf{w}$.

1c. With `w` defined as in 1a, explain the difference between the Maple commands `w[1];`, `w;`, and `print[w];`.

2a. Execute the following sequence of Maple commands, which illustrate how vector functions are defined, evaluated, differentiated, and plotted.

```
# Vector Functions -- Space Curves
restart;
with(linalg):  with(plots):
r:= vector([cos(t),sin(t),t]);  # helix
# To evaluate a vector function at a value of t
subs(t=1,op(r));
# To differentiate a vector function r with respect to t
v:=map(diff,r,t);
# To plot a space curve r
rL:= convert(r,list);
spacecurve(rL,t=-Pi..2*Pi);
```

Insert the plot in your worksheet.

2b. Use the spacecurve command to plot the function `v` over the interval $-\pi \le t \le 2\pi$. Choose the **Boxed** option from the **Axes** menu and use the mouse to rotate the plot until you can see clearly the height on the z axis at which the circle appears. Include this plot in your worksheet and state the exact value of the radius and height of the circle you see.

2c. With `v` defined as in 2a, note the difference in the following Maple commands: `norm(v,2);`, `simplify(norm(v,2));`, `sqrt(dotprod(v,v));`, and `simplify(sqrt(dotprod(v,v));`. Use the simplest expression when doing part (d) of this problem.

2d. The following formulas for the unit tangent `T`, normal `N`, and binormal vectors `B` and the curvature `kappa` can be found in your book:

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}, \qquad \mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}, \qquad \mathbf{B} = \mathbf{T} \times \mathbf{N}, \qquad \kappa = \frac{|d\mathbf{T}/dt|}{|\mathbf{v}|}.$$

Use Maple to define expressions for each of these, using the letters given above, and then simplify these expressions as much as possible. Do not `restart` since you will need the expressions defined in parts 2a and 2c.

2e. Use Maple to verify that the vectors `T`, `N`, and `B` are mutually perpendicular at any value of t.

3. In this problem, we examine a different formula which uses the cross product to compute curvature, and apply it to the case of an ellipse. Although the ellipse is in the xy plane, we define the position vector as a three-dimensional vector so that the cross product formula is applicable.

3a. First define the position vector `r` for an ellipse by executing the following Maple commands.

```
restart;
with(linalg):
r:= vector([c*cos(t), d*sin(t),0]); # ellipse in the plane
```

3b. Now use Maple to define the velocity vector `v` ($\mathbf{v} = d\mathbf{r}/dt$) and the acceleration vector `a` ($\mathbf{a} = d\mathbf{v}/dt$).

3c. Have Maple compute the curvature `kappa` using the formula

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

By looking at the formula you have obtained, answer the following questions.

3d. What is the curvature when $c = d$? (Simplify as much as possible)

3e. For what values of t in $[0, 2\pi)$ is the curvature greatest when $c > d$?

3f. For what values of t in $[0, 2\pi)$ is the curvature greatest when $d > c$?

3g. Verify that your answers are correct in some special cases, by plotting the function κ on the interval $[0, 2\pi)$ first for the choice $c = 3$, $d = 1$ and then for the choice $c = 2$, $d = 4$. Note that one way to do this is to use the Maple command `kappa1:=subs({c=3,d=1},kappa);`.