Mathematics 251: Lab 3 MAXIMUM AND MINIMUM VALUES

In this lab, we use Maple to help visualize and compute the maximum and minimum values of a function of two variables.

Please turn in only the printout of your Maple worksheet. Include explicit answers to all questions asked, using the **text** feature of Maple to insert them in the worksheet (don't write them by hand). Insert graphs directly into your worksheet; instructions were given in the initial handout. Remove from the worksheet any extraneous material such as the output of commands like `with(plots);` and any errors you have made.

1. In this problem we wish to find all the local maxima, local minima, and saddle points of the function

$$f(x,y) = (y^3/9) + 3x^2y + 9x^2 + y^2 + xy + 9;$$

1a. First obtain a rough idea of what this function looks like, by plotting it over the region $-1.5 \le x \le 1.5$, $-7 \le y \le 1$. Use the `axes=BOXED` option so you can see the values of f.

1b. Now use Maple's `diff` and `solve` commands to find all the critical points of f. This can be done by first assigning names such as `fx` and `fy` to the expressions for $\partial f/\partial x$ and $\partial f/\partial y$ obtained by the `diff` command, i.e.,

```
fx:=diff(f,x); fy:=diff(f,y);
```

and then using the `solve` command to obtain the values of x and y for which the equations $\partial f/\partial x = 0$, $\partial f/\partial y = 0$, are satisfied, i.e.,

```
solve({fx=0,fy=0},{x,y});
```

1c. Again using Maple's `diff` command, evaluate f_{xx} and $f_{xx}f_{yy} - (f_{xy})^2$ at each critical point to determine whether it is a local maximum, local minimum, or saddle point. Clearly label in your Maple worksheet the type of each critical point. Note that to evaluate an expression such as `fx` at the point $(1,2)$, one uses the `subs` command, i.e., `subs({x=1,y=2},fx);`

To evaluate both `fx` and `fy` at the point $(1,2)$, use `subs({x=1,y=2},{fx,fy});`

Sometimes the solutions obtained from the `solve` command might be quite long, in which case one would like to avoid retyping them. This may be done by assigning a name to the result, e.g.,

```
soln:= solve({fx=0,fy=0},{x,y});
```

In case there is more than one solution, use the syntax `soln[3]` to refer to the third one. Hence, to evaluate both `fx` and `fy` at `soln[3]`, type

```
subs(soln[3],{fx,fy});
```

1d. For each critical point, obtain a plot of f in a small region around the critical point, which clearly shows whether it is a local maximum, a local minimum, or a saddle point. Make sure it is clear which point each plot is for.

2. Now consider the function $f(x, y) = y(1 - 10xy)\exp(-x^2 - y^2)$. In this case, we would like to find the absolute minimum and absolute maximum of f. Before continuing with Maple, don't forget to use the `restart` command; otherwise many of your variables may already be defined.

2a. First obtain a rough idea of what this function looks like, by plotting it over the region $-5 \le x \le 5$, $-5 \le y \le 5$ and then over the region $-2 \le x \le 2$, $-2 \le y \le 2$.

2b. What happens if you use Maple's `diff` and `solve` commands to try to find the critical points of f?

2c. Now use the `fsolve` command to find the critical points which lie in the region $-2 \le x \le 2$, $-2 \le y \le 2$. There are exactly four of them as indicated by the plot in part (a). The `fsolve` command uses an iterative method (such as Newton's method) to find the roots and sometimes the method does not converge to a root. What happens if you use the command in the form

```
fsolve({fx=0,fy=0},{x,y});
```

where `fx` and `fy` are the names assigned to the expressions obtained for the partial derivatives f_x and f_y, respectively? To improve the chance of convergence, add a region in which you expect the root to lie. You can obtain such a region by using the second of the plots obtained in part (a). For example, one of the critical points can be found by using the following form of the `fsolve` command.

```
p1:= fsolve({fx=0,fy=0},{x,y}, x =-1..0, y=0..1);
```

Note that as in Problem 1, by assigning the name `p1` to the solution of the `fsolve` command, you can then evaluate the function $f(x, y)$ at the values of x and y given by Maple in response to this command by typing `evalf(subs(p1,f));` assuming that `f` is the name assigned to the expression for the function f. The command `evalf` is used to convert the result of the `subs` command to a numerical value.

2d. What are the absolute minimum and absolute maximum values of the function f?

The results of this problem will be used in the next one, so do **not** use the `restart` command before beginning Problem 3.

3. In this problem, we find the absolute minimum and absolute maximum of the function f of Problem 2, but this time restricted to the semicircular region R given by $y \ge 0$, $x^2 + y^2 \le 1.5$. We know from the general theory developed in class (or in the book) that the absolute minimum and maximum of f occur either at critical points of f which lie in R or on the boundary of the region R.

3a. Obtain a plot of f over the region R.

3b. Determine which of the critical points found in Problem 2 lie in the region R and evaluate f at these points. Note that a simple way to check whether the point given by `p1` is inside the circle $x^2 + y^2 = 1.5$ is use the `subs` command to evaluate the expression

$x^2 + y^2 - 1.5$ at `p1`. If this value is less than zero and if the y component of `p1` is ≥ 0, then the point `p1` lies in the region R.

3c. The boundary of R consists of two pieces. The first is the line segment $y = 0$, $-\sqrt{1.5} \leq x \leq \sqrt{1.5}$. What is the value of f on this line segment? Note that you don't need Maple to find it.

3d. The second piece of the boundary of R is the semicircle $-\sqrt{1.5} \leq x \leq \sqrt{1.5}$, $y = \sqrt{1.5 - x^2}$. To find the maximum and minimum of f on the semicircle, set $g(x) = f(x, \sqrt{1.5 - x^2})$ and find the maximum and minimum of g over the interval $-\sqrt{1.5} \leq x \leq \sqrt{1.5}$. To do this, first get a rough idea of what g looks like by using the `plot` command to obtain a plot of the function g over the interval $-\sqrt{1.5} \leq x \leq \sqrt{1.5}$. Once you have identified the approximate location of the absolute maximum and absolute minimum of g from the graph, use the `fsolve` command, specifying a range for x, to find more precise values of $xmax$ and $xmin$, the locations at which the maximum and minimum occur. Then find $ymax$ and $ymin$, the corresponding values of y on the semicircle, and evaluate f at the points $(xmax, ymax)$ and $(xmin, ymin)$.

3e. Now compare the values of f at the critical points inside R (from part (b)), on the line segment $y = 0$ (from part (c)), and at the points $(xmax, ymax)$ and $(xmin, ymin)$ (from part (d)). Clearly state the maximum and minimum values of f over the region R and the coordinates of the points at which they occur.